Classification of 2nd order partial differential equation (PDE)

The general form of 2nd order PDE is

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0 \rightarrow \circledast$$

(where A, B, C are the linear functions of the independent variable x & y or constants)

Elliptic eq: $B^2 - AC < 0$

Parabolic: $B^2 - AC = 0$

Hyperbolic: $B^2 - AC > 0$

Examples

1. $u_{xx} + u_{yy} = 0$; $A = 1$, $C = 1$, $B = 0$

By comparing with $\circledast$

$B^2 - AC = -1 < 0$ (Elliptic)

2. $u_{xx} = \alpha^2 u_t$ (Diffusion Eq)

$A = 1$, $B = 0$, $C = 0$

$B^2 - AC = 0$ (Parabolic Eq).

3. $u_{xx} = \frac{1}{c^2} u_{tt}$

$u_{xx} - \frac{1}{c^2} u_{tt} = 0$

$A = 1$, $B = 0$, $C = -\frac{1}{c^2}$

$B^2 - AC = \frac{1}{c^2} > 0$ (Hyperbolic Eq).

Exercises

Classify the following eqs:

1. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x + 6u_y + 7 = 0$

2. $u_{xx} - 3u_{xy} + \frac{1}{2} u_{yy} + 16 u_y = 0$

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Reduction of 2nd order PDE into Canonical Form

Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0 \quad (1)$$

where A, B, C are functions of x and y and u have continuous first and 2nd order partial derivative. Also, A, B, C do not vanish at the same time.

Let us transform the independent variables by substituting

$$\xi = \xi(x, y)$$
$$\eta = \eta(x, y)$$

ξ and η are such that

$$\frac{D(\xi, \eta)}{D(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

In the new coordinate (ξ, η) (1) can be written as

$$\bar{A} \frac{\partial^2 u}{\partial \xi^2} + 2\bar{B} \frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C} \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, u_\xi, u_\eta) = 0,$$

where

$$\bar{A} = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$\bar{C} = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2,$$

$$\bar{B} = A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x}\right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}.$$

(3)

Aside

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right)$$

$$= \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$\Rightarrow$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}}$$

(2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial y}\right)$$

(4)

$$\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \xi}{\partial y}$$

$$+ \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]\frac{\partial \eta}{\partial y}$$

$$+ \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y}$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$= \boxed{\begin{aligned}&\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 u}{\partial \xi \partial \eta}\\ &+ \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}\end{aligned}} \rightarrow (3)$$

Now

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y}$$

$$+ \frac{\partial u}{\partial \eta}\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y}\right)$$

$$= \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)\frac{\partial \eta}{\partial y}\right]\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} \times$$

$$\times \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y}\right]\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}$$

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⑤

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial y^2}$$

or

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}} \rightarrow (4)$$

Using (2), (3) and (4) in (1) we have

$$\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}\right] + 2B\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right)\frac{\partial^2 u}{\partial \xi \partial \eta}\right.$$

$$+ \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y}$$

$$\left. + \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial x \partial y}\right]$$

$$+ C\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\right] +$$

$$+ F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$

$$= \left[A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial \xi^2} +$$

(P.T.O)

$$+2\left[A\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]\frac{\partial^2u}{\partial\xi\partial\eta}$$

$$+\left[A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2\right]\frac{\partial^2u}{\partial\eta^2}$$

$$+F\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0$$

or

$$\boxed{\bar{A}\frac{\partial^2u}{\partial\xi^2}+2\bar{B}\frac{\partial^2u}{\partial\xi\partial\eta}+\bar{C}\frac{\partial^2u}{\partial\eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)=0} \quad (5)$$

where

$$\left.\begin{aligned}
\bar{A}&=A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2,\\
\bar{B}&=A\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}\right)\\
&\quad+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y},\\
\bar{C}&=A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2
\end{aligned}\right\} \quad (6)$$

Note that

$$\bar{B}^2-\bar{A}\bar{C}=(B^2-AC)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2 \quad (7)$$

where A, B, C are not zero at the same time. Thus the transformation of independent variables does not change the type of equation.

Since

$$\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} = \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix} \neq 0$$

So $\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2$ is a positive expression.

$\boxed{\text{Show that}}$ $\quad \bar{B}^2 - \bar{A}\bar{C} = (B^2 - AC)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2.$

Aside From (6)

$$\bar{B}^2 - \bar{A}\bar{C} = \left[A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]^2$$

$$- \left[A\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C\left(\frac{\partial\xi}{\partial y}\right)^2\right]$$

$$\times\left[A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + C\left(\frac{\partial\eta}{\partial y}\right)^2\right].$$

$$\bar{B}^2 - \bar{A}\bar{C} = \cancel{A^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2} + B^2\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2$$

$$+ \cancel{C^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2} + 2AB\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)$$

$$+ 2BC\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)$$

$$+ 2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} - \cancel{A^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2}$$

$$- 2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} - AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2 - 2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}$$

(5)

$$-4B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}-2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}$$

$$-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2-2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$\cancel{-C^2\left(\frac{\partial\eta}{\partial y}\right)^2\left(\frac{\partial\xi}{\partial y}\right)^2}$$

$$=B^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+B^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$+2B^2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\cancel{+2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}$$

$$\cancel{+2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}\cancel{+2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}$$

$$\cancel{+2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}+2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$\cancel{-2AB\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}-AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$\cancel{-2AB\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial x}}-4B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$\cancel{-2BC\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}}-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$\cancel{-2BC\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}}$$

or

$$\bar{B}^2-\bar{A}\bar{C}=B^2\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2+B^2\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$-2B^2\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+2AC\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}$$

$$-AC\left(\frac{\partial\xi}{\partial x}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2-AC\left(\frac{\partial\xi}{\partial y}\right)^2\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$= B^2\left[\left(\frac{\partial \xi}{\partial x}\right)^2\left(\frac{\partial \eta}{\partial y}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\left(\frac{\partial \eta}{\partial x}\right)^2 - 2\,\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right]$$

$$- AC\left[\left(\frac{\partial \xi}{\partial x}\right)^2\left(\frac{\partial \eta}{\partial y}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\left(\frac{\partial \eta}{\partial x}\right)^2 - 2\,\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right]$$

$$= (B^2 - AC)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2$$

Now, consider

$$\boxed{A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2B\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right) + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0} \quad (8)$$

Case I **Hyperbolic Equation.**

$B^2 - AC > 0$

Consider that either $A \neq 0$ or $C \neq 0$, Eq. (8) can be written as

$$(\phi_x)^2 + \frac{2B}{A}\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right) + \frac{C}{A}\left(\frac{\partial \phi}{\partial y}\right)^2 = 0.$$

This is quadratic in ϕ_x and ϕ_y. Consider it is quadratic in ϕ_x than we have by Formula

$$\phi_x = \frac{-B\phi_y \pm \phi_y\sqrt{B^2-AC}}{A}$$

or

$$A\phi_x = \left(-B \pm \sqrt{B^2-AC}\right)\phi_y$$

or

$$\boxed{A\frac{\partial\phi}{\partial x} + \left(B \pm \sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y} = 0} \longrightarrow (9)$$

Thence $\left[A\frac{\partial\phi}{\partial x} + \left(B+\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y}\right]$ and $\left[A\frac{\partial\phi}{\partial x} + \left(B-\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y}\right]$ are factors of (8).

so (8) can be written as

$$\left[A\frac{\partial\phi}{\partial x} + \left(B+\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y}\right]\left[A\frac{\partial\phi}{\partial x} + \left(B-\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y}\right] = 0. \longrightarrow (10)$$

From above eq.

$$A\frac{\partial\phi}{\partial x} + \left(B+\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y} = 0, \longrightarrow (11)$$

$$A\frac{\partial\phi}{\partial x} + \left(B-\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y} = 0. \longrightarrow (12)$$

Note that solutions of (11) and (12) will be the solution of (8).

To solve (11) and (12) the corresponding auxillary equations are

$$\frac{dx}{A} = \frac{dy}{B+\sqrt{B^2-AC}} \longrightarrow (13)$$

$$\frac{dx}{A} = \frac{dy}{B-\sqrt{B^2-AC}} \longrightarrow (14)$$

From (13) and (14), it follows that

$$\boxed{\begin{aligned} A\,dy - \left(B+\sqrt{B^2-AC}\right)dx = 0 \\ A\,dy - \left(B-\sqrt{B^2-AC}\right)dx = 0 \end{aligned}} \longrightarrow (15)$$

Solutions of (15) are

$$\boxed{\begin{aligned} \phi_1(x,y) &= \text{Constant} \\ \phi_2(x,y) &= \text{constant.} \end{aligned}} \longrightarrow (16)$$

Let $\xi = \xi(x,y) = \phi_1(x,y)$

$\eta = \eta(x,y) = \phi_2(x,y)$.

[ϕ_1 and ϕ_2 are the solution of (8) so ξ and η are also solutions of (8)] so

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2 = 0$$

$\Rightarrow \boxed{\bar{A} = 0}$ using (6)

And

$$A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + C\left(\frac{\partial \eta}{\partial y}\right)^2 = 0$$

i.e. $\boxed{\bar{C} = 0}$ using (6)

So the transformed eq. (5) becomes

$$2\bar{B}\frac{\partial^2 u}{\partial \xi \partial \eta} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)}$$ (canonical form for hyperbolic equation)

Ex $y^2 u_{xx} - x^2 u_{yy} = 0 \longrightarrow (*)$

$A = y^2, \quad B = 0, \quad C = -x^2$

$B^2 - AC = x^2 y^2 > 0$ (Hyperbolic Eq)

Now ξ and η are the solutions of

$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (1)$

Making use of A, B and C in (1) we have

$y^2 dy \mp \sqrt{y^2 x^2}\, dx = 0$

$y^2 dy \mp xy\, dx = 0$

$y\, dy \mp x\, dx = 0$

Integ

$\frac{y^2}{2} - \frac{x^2}{2} = c_1, \quad \frac{y^2}{2} + \frac{x^2}{2} = c_2.$

$\xi = \phi_1, \quad \eta = \phi_2$

So $\xi = \frac{y^2 - x^2}{2} \to (2), \quad \eta = \frac{y^2 + x^2}{2} \to (3)$

From (2) and (3)

$\xi_x = -x, \quad \xi_y = y$	$\eta_x = x, \quad \eta_y = y$
$\xi_{xx} = -1, \quad \xi_{yy} = 1$	$\eta_{xx} = 1, \quad \eta_{yy} = 1$

$\to (4)$

Now

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}(\eta_x)^2$$
$$+ u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}$$

Using (4) we have

$$\boxed{u_{xx} = x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta} \longrightarrow (5)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}(\eta_y)^2$$
$$+ u_\xi\,\xi_{yy} + u_\eta\,\eta_{yy}$$

Using (4) we obtain

$$\boxed{u_{yy} = y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta} \longrightarrow (6)$$

Using (5) and (6) in (*) we have

$$y^2(x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta)$$
$$-x^2(y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta) = 0$$

$$\cancel{x^2y^2 u_{\xi\xi}} - 2x^2y^2 u_{\xi\eta} + x^2y^2 u_{\eta\eta} - y^2 u_\xi$$
$$+ y^2 u_\eta - \cancel{x^2y^2 u_{\xi\xi}} - 2x^2y^2 u_{\xi\eta} - x^2y^2 u_{\eta\eta}$$
$$- x^2 u_\xi - x^2 u_\eta = 0$$

$$-4x^2y^2 u_{\xi\eta} - y^2 u_\xi - x^2 u_\xi + y^2 u_\eta - x^2 u_\eta = 0 \longrightarrow (7)$$

From (2) and (3)

$\xi + \eta = y^2, \quad \eta - \xi = x^2$

using in (7) we get

$$-4(\eta^2 - \xi^2) u_{\xi\eta} - 2\eta u_\xi + 2\xi u_\eta - \cancel{\xi u_\xi} + \cancel{\xi u_\xi}$$
$$+ \cancel{\eta u_\eta} - \eta u_\eta = 0.$$

$$\Rightarrow -4(\eta^2 - \xi^2) u_{\xi\eta} - 2\eta u_\xi + 2\xi u_\eta = 0$$

or $2(\xi^2 - \eta^2) u_{\xi\eta} - \eta u_\xi + \xi u_\eta = 0$

or $$\boxed{u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)} u_\eta}$$

Example: $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2 \longrightarrow (1)$

$A = 4, \quad B = \frac{5}{2}, \quad C = 1$

$B^2 - AC = \frac{9}{4} > 0$ Hyperbolic Eq.

$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$

Using values of A, B, C in (2) we get.

$4dy - 4dx = 0 \Rightarrow dy - dx = 0 \longrightarrow (3)$

$4dy - dx = 0 \longrightarrow (4)$

Integ (3) and (4)

$y - x = c_1$

$4y - x = c_2$

so $$\boxed{\begin{aligned} \xi &= y - x \\ \eta &= 4y - x \end{aligned}} \longrightarrow (5)$$

$$\xi_x = -1,\ \xi_{xx} = 0,\ \xi_y = 1,\ \xi_{yy} = 0,\ \xi_{xy} = 0,$$
$$\eta_x = -1,\ \eta_{xx} = 0,\ \eta_y = 4,\ \eta_{yy} = 0,\ \eta_{xy} = 0 \longrightarrow (6)$$

Using (6) we get

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$\boxed{u_x = -u_\xi - u_\eta} \longrightarrow (7)$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$\boxed{u_y = u_\xi + 4u_\eta} \longrightarrow (8)$$

and

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

using (6) we get

$$u_{xx} = u_{\xi\xi}(-1)^2 + 2u_{\xi\eta}(-1)(-1) + u_{\eta\eta}(-1)^2 + u_\xi(0) + u_\eta(0)$$

or

$$\boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (9)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \eta_x\xi_y)u_{\xi\eta} + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_y\eta_{xy}$$

or $= u_{\xi\xi}(-1)(1) + (-5)u_{\xi\eta} + u_{\eta\eta}(-1)(4) + u_\xi(0) + u_\eta(0)$

$$\boxed{u_{xy} = -u_{\xi\xi} - 5u_{\xi\eta} - 4u_{\eta\eta}} \rightarrow (10)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$= u_{\xi\xi}(1)^2 + 2u_{\xi\eta}(1)(4) + u_{\eta\eta}(4)^2 + u_\xi(0) + u_\eta(0)$$

$$= \boxed{u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta}} \rightarrow (11)$$

Using (7) to (11) in (1) we get.

$$4[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}] + 5[-u_{\xi\xi} - 5u_{\xi\eta} - 4u_{\eta\eta}] + u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta} - u_\xi - u_\eta + u_\xi + 4u_\eta = 2$$

$$\Rightarrow 4u_{\xi\xi} + 8u_{\xi\eta} + 4u_{\eta\eta} - 5u_{\xi\xi} - 25u_{\xi\eta} - 20u_{\eta\eta} + u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta} - u_\xi - u_\eta + u_\xi + 4u_\eta = 2$$

$$\Rightarrow -9u_{\xi\eta} + 3u_\eta = 2$$

$$3u_{\xi\eta} - u_\eta = -\frac{2}{3} \Rightarrow \boxed{u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{2}{9}}$$

Elliptic Equation $(B^2 - AC < 0)$.

$$A\,dy - (B + \sqrt{B^2 - AC})\,dx = 0 \quad \text{——— (1)}$$

$\phi(x, y) = c$ is a soln of (1)

$$\Rightarrow \phi_1(x, y) + i\,\phi_2(x, y) = c$$

($\sqrt{B^2 - AC}$ is imaginary for this case)

$$\xi = \xi(x,y) = \phi_1(x,y)$$
$$\eta = \eta(x,y) = \phi_2(x,y)$$
$$\phi = \phi_1 + i\phi_2 = \xi + i\eta \longrightarrow (2)$$

Now as ξ and η are the solutions of

$$A\left(\frac{\partial\phi}{\partial x}\right)^2 + 2B\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right) + C\left(\frac{\partial\phi}{\partial y}\right)^2 = 0 \longrightarrow (3)$$

Using (2) in (3) we get

$$A\left[\frac{\partial}{\partial x}(\xi+i\eta)\right]^2 + 2B\left[\frac{\partial}{\partial x}(\xi+i\eta)\frac{\partial}{\partial y}(\xi+i\eta)\right]$$
$$+ C\left[\frac{\partial}{\partial y}(\xi+i\eta)\right]^2 = 0$$

$$A\left[\frac{\partial\xi}{\partial x} + i\frac{\partial\eta}{\partial x}\right]^2 + 2B\left(\frac{\partial\xi}{\partial x} + i\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\xi}{\partial y} + i\frac{\partial\eta}{\partial y}\right)$$
$$+ C\left[\frac{\partial\xi}{\partial y} + i\frac{\partial\eta}{\partial y}\right]^2 = 0.$$

or

$$A\left[\left(\frac{\partial\xi}{\partial x}\right)^2 + i^2\left(\frac{\partial\eta}{\partial x}\right)^2 + 2i\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\right]$$
$$+2B\left[\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + i\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + i\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y} + i^2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\right]$$
$$+C\left[\left(\frac{\partial\xi}{\partial y}\right)^2 + i^2\left(\frac{\partial\eta}{\partial y}\right)^2 + 2i\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right] = 0.$$

or.

(18)

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 - A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2iA\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}$$
$$+2iB\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + 2iB\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}$$
$$+C\left(\frac{\partial \xi}{\partial y}\right)^2 - C\left(\frac{\partial \eta}{\partial y}\right)^2 + 2iC\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0 + i0$$

Equating real and imaginary parts we have.

(i.e. real part gives)

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 - A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} - 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}$$
$$+C\left(\frac{\partial \xi}{\partial y}\right)^2 - C\left(\frac{\partial \eta}{\partial y}\right)^2 = 0$$

or

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2$$

i.e. $\boxed{\bar{A} = \bar{C}}$

(Imaginary part gives)

$$2A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + 2B\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}$$
$$+2C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0$$

$$A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) = 0$$

$$\Longrightarrow \boxed{\bar{B} = 0}$$

So the transformed equation is reduced to

$$\boxed{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)}$$

Ex $u_{xx} + x^2 u_{yy} = 0$ —— (1)

$A = 1,\ B = 0,\ C = x^2$

$B^2 - AC = -x^2 < 0$

Elliptic eq.

Now ξ and η are respectively the real and imaginary parts of the solution of the eq.

$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0$

Using values of A, B and C we have

$dy - \sqrt{-x^2}\,dx = 0$

$dy \pm i x\,dx = 0$

Integ $y \pm i\frac{x^2}{2} = c$ as $c = \phi_1 + i\phi_2$

so $\xi = \phi_1$, $\eta = \phi_2$

$\Rightarrow \boxed{\xi = y,\ \eta = \frac{x^2}{2}}$ —— (2)

$\xi_x = 0,\ \xi_{xx} = 0,\ \xi_y = 1,\ \xi_{yy} = 0$

$\eta_x = x,\ \eta_{xx} = 1,\ \eta_y = 0,\ \eta_{yy} = 0$ —— (3)

As

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}.$$

using (3) we obtain

$$u_{xx} = x^2 u_{\eta\eta} + u_\eta \longrightarrow (4)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y \eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$\Rightarrow \boxed{u_{yy} = u_{\xi\xi}} \longrightarrow (5)$$

Using (4) and (5) in (1) we get

$$x^2 u_{\eta\eta} + u_\eta + x^2 u_{\xi\xi} = 0 \longrightarrow (6)$$

From (2)

$$x^2 = 2\eta$$

Using in (6) we obtain

$$2\eta u_{\eta\eta} + u_\eta + 2\eta u_{\xi\xi} = 0$$

$$2\eta(u_{\eta\eta} + u_{\xi\xi}) = -u_\eta$$

or $\boxed{u_{\eta\eta} + u_{\xi\xi} = -\frac{1}{2\eta} u_\eta}$

III **Parabolic Equation**

$$B^2 - AC = 0 \longrightarrow (1)$$

$$A\,dy - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$$

With (1), (2) becomes

$$A\,dy - B\,dx = 0$$

which has the solution $\phi(x,y) = c$

we choose $\xi = \phi(x,y) \longrightarrow (3)$

and η arbitrary.

Now ξ is a solution of

$$A\left(\frac{\partial\phi}{\partial x}\right)^2+2B\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)+C\left(\frac{\partial\phi}{\partial y}\right)^2=0 \longrightarrow (4)$$

With (3) in (4) to arrive at

$$A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\xi}{\partial y}\right)+C\left(\frac{\partial\xi}{\partial y}\right)^2=0$$

Or $\boxed{\bar{A}=0} \longrightarrow (5)$

As we know

$$A\frac{\partial\phi}{\partial x}+\left(B+\sqrt{B^2-AC}\right)\frac{\partial\phi}{\partial y}=0 \longrightarrow (5a)$$

$B^2-AC=0$

$$A\frac{\partial\xi}{\partial x}+B\frac{\partial\xi}{\partial y}=0 \longrightarrow (6)$$

(With (3) in (5a)).

$\Rightarrow$

$$AB\frac{\partial\xi}{\partial x}+B^2\frac{\partial\xi}{\partial y}=0$$

$$AB\frac{\partial\xi}{\partial x}+AC\frac{\partial\xi}{\partial y}=0$$

$(A\neq 0)$.

$$\boxed{B\frac{\partial\xi}{\partial x}+C\frac{\partial\xi}{\partial y}=0} \longrightarrow (7)$$

As

$$\bar{B}=A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}$$

$$=\frac{\partial\eta}{\partial x}\underbrace{\left(A\frac{\partial\xi}{\partial x}+B\frac{\partial\xi}{\partial y}\right)}_{\to 0}+\frac{\partial\eta}{\partial y}\underbrace{\left(B\frac{\partial\xi}{\partial x}+C\frac{\partial\xi}{\partial y}\right)}_{\to 0}$$

$\Rightarrow \boxed{\bar{B}=0} \longrightarrow (8)$ (With (7)).

So we get the canonical form as

$$\boxed{\frac{\partial^2 u}{\partial\eta^2}=F_3\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)}$$

Example:
$$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0 \longrightarrow (1)$$

$A = 1,\ B = -x,\ C = x^2$

$B^2 - AC = 0$ (Parabolic Eq.)

$A\,dy - B\,dx = 0$

using values, to obtain

$$dy + x\,dx = 0$$

Integrating

$$y + \frac{x^2}{2} = c$$

So $\boxed{\xi = y + \frac{x^2}{2}}$

and $\eta = x$ arbitrary.

$\xi_x = x,\ \xi_{xx} = 1,\ \xi_{xy} = 0$

$\xi_y = 1,\ \xi_{yy} = 0,\ \xi_{xy} = 0$

$\eta_x = 1,\ \eta_{xx} = 0,\ \eta_{xy} = 0$

$\eta_y = 0,\ \eta_{yy} = 0,\ \eta_{xy} = 0$

So we have

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= u_\xi + u_\eta (0)$$

$$\Rightarrow \boxed{u_y = u_\xi} \longrightarrow (2)$$

and

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$\Rightarrow \boxed{u_{xx} = x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta} + u_\xi} \longrightarrow (3)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$\Rightarrow u_{yy} = u_{\xi\xi} + 2u_{\xi\eta}(0)$$

$$\Rightarrow \boxed{u_{yy} = u_{\xi\xi}} \longrightarrow (4)$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \eta_x\xi_y)u_{\xi\eta} + \eta_x\eta_y u_{\eta\eta} + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

or

$$\boxed{u_{xy} = x u_{\xi\xi} + u_{\xi\eta}} \longrightarrow (5)$$

Using all these in (1) to obtain

$$x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta} + u_\xi - 2x(x u_{\xi\xi} + u_{\xi\eta}) + x^2(u_{\xi\xi}) - 2u_\xi = 0$$

$$\cancel{x^2 u_{\xi\xi}} + \cancel{2x u_{\xi\eta}} + u_{\eta\eta} + u_\xi - \cancel{2x^2 u_{\xi\xi}} - \cancel{2x u_{\xi\eta}} + \cancel{x^2 u_{\xi\xi}} - 2u_\xi = 0$$

$$\Rightarrow \boxed{u_{\eta\eta} - u_\xi = 0}$$

Ex $x^2 u_{xx} + 2u_{xy}(xy) + y^2 u_{yy} = 0 \longrightarrow (1)$

$A = x^2,\ B = xy,\ C = y^2$

$B^2 - AC = 0$ (Parabolic Eq.)

$A\,dy - B\,dx = 0$

$x^2\,dy - xy\,dx = 0$

$\frac{dy}{y} = \frac{dx}{x}$ Integ.

$\therefore \ln\frac{y}{x} = \ln c_1$

$\Rightarrow \frac{y}{x} = c_1$,

$$\boxed{\begin{aligned} \xi(x,y) &= y/x \\ \eta &= y \end{aligned}} \longrightarrow (2)$$

Also

$\xi_x = -\frac{y}{x^2}$, $\xi_{xx} = \frac{2y}{x^3}$, $\xi_y = \frac{1}{x}$, $\xi_{yy} = 0$,

$\xi_{xy} = -\frac{1}{x^2}$, $\eta_x = 0$, $\eta_{xx} = 0$, $\eta_y = 1$, $\eta_{yy} = 0$

$\eta_{xy} = 0$

Now

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_{\eta\eta}(\eta_x)^2 + u_\eta \eta_{xx}$$

Using values we get

$$\boxed{u_{xx} = \frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi} \longrightarrow (3)$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi \xi_{yy} + u_{\eta\eta}(\eta_y)^2 + u_\eta \eta_{yy}$$

or

$$\boxed{u_{yy} = \frac{u_{\xi\xi}}{x^2} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (4)$$

(25)

and

$$u_{xy} = u_{\xi\xi}\,\xi_x\,\xi_y + (\xi_x\,\eta_y + \eta_x\,\xi_y)\,u_{\xi\eta} + u_{\eta\eta}\,\eta_x\,\eta_y + u_\xi\,\xi_{xy} + u_\eta\,\eta_{xy}$$

or

$$\boxed{u_{xy} = \frac{-y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi} \longrightarrow (5)$$

Using the values in (1), we have

$$x^2\left(\frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi\right) + 2xy\left(\frac{-y}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\xi\right)$$

$$+ y^2\left(\frac{1}{x^2}u_{\xi\xi} + \frac{2}{x}u_{\xi\eta} + u_{\eta\eta}\right) = 0$$

$\Longrightarrow$

$$\cancel{\frac{y^2}{x^2}u_{\xi\xi}} + \cancel{\frac{2y}{x}u_\xi} - \cancel{\frac{2y^2}{x^2}u_{\xi\xi}} - \cancel{\frac{2y^2}{x}u_{\xi\eta}} - \cancel{\frac{2y}{x}u_\xi}$$

$$+ \cancel{\frac{y^2}{x^2}u_{\xi\xi}} + \cancel{\frac{2y^2}{x}u_{\xi\eta}} + y^2u_{\eta\eta} = 0$$

$$\Longrightarrow y^2\,u_{\eta\eta} = 0$$

$$\Longrightarrow \boxed{u_{\eta\eta} = 0}$$

Integ.

$$u_\eta = f(\xi)$$

Again integ.

$$u = \int f(\xi)\,d\eta + g(\xi)$$

$$= \eta\,f(\xi) + g(\xi)$$

$$\boxed{u = y\,f(y/x) + g(y/x)}.$$

Exercise 3

Classify the following equations:

1. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x + 6y + 7 = 0$
2. $8u_{xx} - 8u_{xy} + 2u_{yy} + 17u_x - 13u = 0$
3. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0.$

Reduce to canonical form:

1. $y\dfrac{\partial^2 z}{\partial x^2} + (x+y)\dfrac{\partial^2 z}{\partial x \partial y} + x\dfrac{\partial^2 z}{\partial y^2} = 0,$
2. $y^2\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x \partial y} + x^2\dfrac{\partial^2 z}{\partial y^2} - \dfrac{y^2}{x}\dfrac{\partial z}{\partial x} - \dfrac{x^2}{y}\dfrac{\partial z}{\partial y} = 0$
3. $\dfrac{\partial^2 z}{\partial x^2} + x^2\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{x}\dfrac{\partial z}{\partial x} = 0$
4. $\dfrac{\partial^2 z}{\partial x^2} - x^2\dfrac{\partial^2 z}{\partial y^2} = 0$
5. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0.$
6. $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 4x^2,$
7. $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0$
8. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
9. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
10. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0.$

Ex Reduce into the canonical form and hence find the general solution

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

with Cauchy's data

$$u(x,0) = f(x),$$

$$u_t(x,0) = g(x).$$

soln $u_{xx} - \frac{1}{c^2} u_{tt} = 0 \longrightarrow (1)$

$A = 1, \quad B = 0, \quad C = -\frac{1}{c^2}$

$\therefore B^2 - AC = \frac{1}{c^2} > 0$ (Hyperbolic Eq).

$A\,dt - (B \pm \sqrt{B^2 - AC})\,dx = 0 \longrightarrow (2)$

With A, B and C, the above equations yield

$dt - \frac{1}{c} dx = 0 \longrightarrow (3)$

$dt + \frac{1}{c} dx = 0 \longrightarrow (4)$

Integ (4) we get

$t - \frac{x}{c} = k_1^*$

$ct - x = c k_1^*$

$x - ct = k_1, \quad k_1 = -ck_1^* \longrightarrow (5)$

Similarly, integ (4) we get.

$x + ct = k_2, \quad k_2 = -ck_2^* \longrightarrow (6)$

so
$$\boxed{\begin{aligned} \xi(x,t) &= x + ct \\ \eta(x,t) &= x - ct \end{aligned}} \longrightarrow (7)$$

$$\xi_x = 1, \; \xi_{xx} = 0, \; \xi_t = c, \; \xi_{tt} = 0, \; \xi_{xt} = 0$$

$$\eta_x = 1, \; \eta_{xx} = 0, \; \eta_t = -c, \; \eta_{tt} = 0, \; \eta_{xt} = 0$$

$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_{\eta\eta}(\eta_x)^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(1)^2 + 2u_{\xi\eta}(1)(1) + u_\xi(0) + u_{\eta\eta}(1)^2 + u_\eta(0)$$

$$\Rightarrow \boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \longrightarrow (8)$$

$$u_{tt} = u_{\xi\xi}(\xi_t)^2 + 2u_{\xi\eta}\xi_t\eta_t + u_\xi \xi_{tt} + u_{\eta\eta}(\eta_t) + u_\eta \eta_{tt}$$

$$= u_{\xi\xi}c^2 + 2u_{\xi\eta}(c)(-c) + u_\xi(0) + u_{\eta\eta}(-c)^2 + u_\eta(0)$$

or

$$\boxed{u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}} \longrightarrow (9)$$

Using (8) and (9) in (1) we have

$$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - \frac{1}{c^2}\left(c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}\right) = 0$$

or

$$\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} + 2u_{\xi\eta} - \cancel{u_{\eta\eta}} = 0$$

$$\Rightarrow 4u_{\xi\eta} = 0$$

$4 \neq 0$ so $\boxed{u_{\xi\eta} = 0} \longrightarrow (10)$ (Required canonical form)

Integ. (10) w.r.t η we get

$$u_\xi = f(\xi)$$

Integ w.r.t ξ we have

$$u = \int f(\xi)\,d\xi + f(\eta)$$

or $$u = \phi(\xi) + \psi(\eta)$$

$$\boxed{u_{(x,t)} = \phi(x+ct) + \psi(x-ct)} \longrightarrow (11)$$

(General soln of (1))

b.cs
$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\}$$

From (11)

$$u(x,0) = \phi(x) + \psi(x) = f(x) \longrightarrow (12)$$

$$u_t(x,t) = c\,\phi'(x+ct) - c\,\psi'(x-ct)$$

$$u_t(x,0) = c\,\phi'(x) - c\,\psi'(x) = g(x) \longrightarrow (13)$$

From (12)

$$\phi'(x) + \psi'(x) = f'(x) \longrightarrow (14)$$

Xing (14) by c we get

$$c\,\phi'(x) + c\,\psi'(x) = c f'(x) \longrightarrow (15)$$

Adding (13) and (15) we obtain

$$g(x) + c\, f'(x) = 2c\, \phi'(x).$$

or $$\boxed{\phi'(x) = \frac{1}{2c} g(x) + \frac{1}{2} f(x)} \rightarrow (16)$$

Subtracting (13) from (15) to get

$$c\, f'(x) - g(x) = 2c\, \psi'(x)$$

$$\Rightarrow \psi'(x) = \frac{1}{2c}\, c\, f'(x) - \frac{1}{2c} g(x)$$

or $$\boxed{\psi'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x)} \rightarrow (17)$$

Integrating (16) and (17) we have

$$\phi(x) = \frac{1}{2c}\int g(x)\,dx + \frac{1}{2} f(x) + k_3 \rightarrow (18)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int g(x)\,dx + k_4 \rightarrow (19)$$

+ing (18) and (19)

$$\phi(x) + \psi(x) = f(x) + k_3 + k_4 \rightarrow (20)$$

From (12)

$$\phi(x) + \psi(x) = f(x)$$

so, (20) yields

$$f(x) = f(x) + k_3 + k_4 \Rightarrow k_3 + k_4 = 0$$
$$\Rightarrow k_3 = -k_4 = k \text{ (say)}$$

Using in (18) and (19) we have

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int g(x)\, dx + k$$

or

$$\boxed{\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(z)\, dz + k} \rightarrow (21)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int g(x)\, dx - k$$

$$\boxed{\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(z)\, dz - k} \rightarrow (22)$$

From (21) and (22)

$$\boxed{\begin{aligned} \phi(x+ct) &= \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(z)\, dz + k \\ \psi(x-ct) &= \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(z)\, dz - k \end{aligned}} \rightarrow (23)$$

Using (23) in (11) to obtain

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_0^{x+ct} g(z)\, dz + \cancel{k} - \frac{1}{2c} \int_0^{x-ct} g(z)\, dz \cancel{-k}$$

or

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_0^{x+ct} g(z)\, dz - \frac{1}{2c} \int_0^{x-ct} g(z)\, dz$$

or

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz + \frac{1}{2c}\int_{x-ct}^{0} g(z)\,dz$$

or

$$\boxed{u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz.}$$

This is called D' Alembert solution.

14/9/03

Mathematical Modeling

Mathematical Modeling is the process in which the evolution or the state of a real-life system is represented by a set of mathematical relations, after proper approximations and idealizations. This process is in general divided into six steps:

1. Objective
2. Background
3. Approximations and idealizations
4. Modeling
5. Model validation
6. Compounding.

1. Objective

Here the real-life system to be modeled is defined.

2. Background

Here the pertinent laws and data about the system must be examined. In particular if no data are available we must carry out proper experiments to obtain this information. Thence we should be able to identify the important variables that influence the evolution of the system and their relations.

3. Approximations and idealizations

Even when constructing a prototype model, some approximations and idealizations of reality must be made. Thus, all mathematical models are approximations of reality to some extent. These approximations place certain limitations on the validity of mathematical model and its correlation with the actual behavior of the system.

4. Modeling

At this stag the mathematical relations that govern the behavior of the system are derived.

5. Model Validation

Methods must be devised to solve the model equations and compare their predictions with the actual data about the system. If large unaccountable deviations between the model predictions and data are detected, the model must be reexamined and modified accordingly.

6. Compounding

At this stage the prototype model is modified to take into account some aspects of the system that were neglected earlier in order to simplify the modeling process.

Applications:

The Continuity Equation

Objective

Derive a model equation for the traffic flow on a highway without exits and with one entrance and one lane.

Discussion

One possible approach to modelling the traffic flow is to describe each car as a finite element on the highway and then write a discrete model, which describes the motion of each such car. However, if there are many cars on the highway, this approach is not practical and it is better to construct a continuous model, which treats these cars as "smeared out" quantities. We construct such a continuous model here.

Approximations and Idealizations

1. We assume that the highway is infinite
2. We define the car density $\rho(x,t)$ as

$$\rho(x,t) \simeq \frac{\text{Number of cars on the interval } [x, x+\Delta x] \text{ at time } t}{\Delta x}$$

where Δx must be large compared to a car length. [Otherwise, $\rho(x,t)=1$ if there is a car at x in time t, or $\rho(x,t)=0$ if there is none.] ~~Hwlfbdf~~

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3. We assume that there are no accidents on the highway (or that their number is negligible). Hence, we can formulate the principle of car conservation (which is equivalent to that of mass conservation) as follows:

"The rate at which the number of cars on the segment $[a,b]$ is changing equals the rate at which they enter less the rate at which they are leaving it".

4. We define the car flux q per lane as

$$q(x,t) = \frac{\text{Cars}}{\text{Lane}\cdot\text{Time}}$$

$$= \rho(x,t)\,u(x,t). \longrightarrow (1)$$

where $u(x,t)$ is the car's speed at x at time t; ~~at~~

Modeling

To derive a model equation for the traffic flow, consider a finite section of the road between a and b.

The number n of cars in this segment at time t is

$$n(t,a,b) = \int_a^b \rho(x,t)\,dx.$$

The rate of change in this quantity is

$$\frac{dn}{dt} = \int_a^b \frac{\partial \rho(x,t)}{\partial t}\, dx \longrightarrow (2)$$

This rate of change must equal the flux of cars entering at a less the flux of cars leaving at b (Remember that the flux was defined per lane) Therefore

$$\frac{dn}{dt} = q(a,t) - q(b,t) = -\int_a^b \frac{\partial q(x,t)}{\partial x}\, dx. \longrightarrow (3)$$

From Eqs. (2) and (3)

$$\int_a^b \frac{\partial \rho}{\partial t}\, dx = -\int_a^b \frac{\partial q}{\partial x}\, dx$$

$$\Longrightarrow \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x}\right] dx = 0$$

Since a and b are arbitrary, it follows that

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

using (1) we have

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0} \longrightarrow (4)$$

This is the continuity equation in one dimension. Notice, however, that this equation contains two unknown quantities, ρ and u. Therefore, to solve it we must either be able to express $u = u(\rho)$ or find an additional equation that relates these two quantities.

Objective

Derive a prototype model equation for the voltage and current in a long, uniform two-wire transmission line.

Background

The most common forms of transmission lines are coaxial and two wire types. The coaxial transmission line consists of two concentric circular cylinders of metals. The two wire-types consists of two parallel wires, one of which is used as ground.

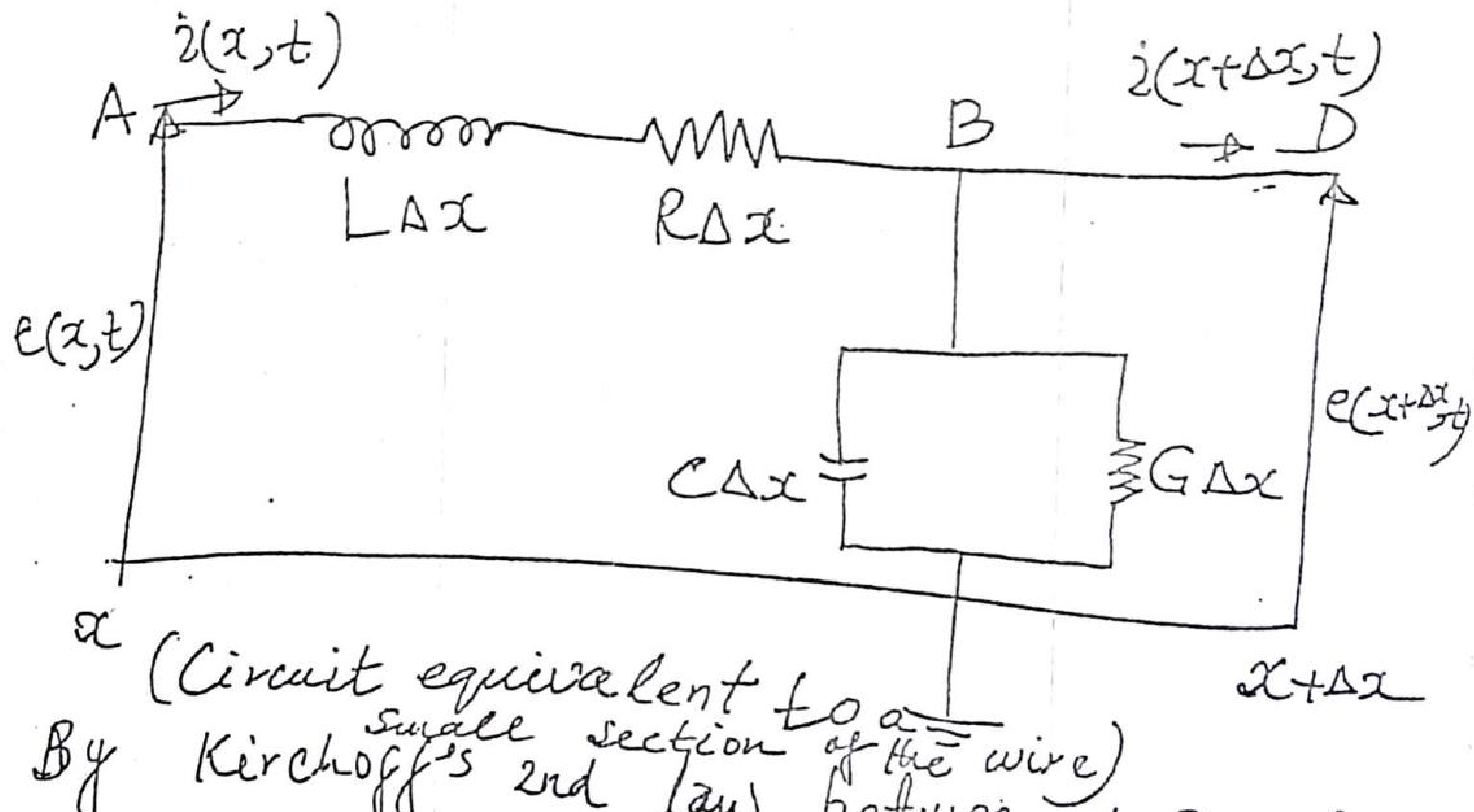
The passage of an electric current through a cable always involves a leakage, which leads to a loss of electrical energy. For short distances this loss can usually be ignored. However, over long distances, which are found in transmission lines, these losses must be taken into account.

Modeling

Since the transmission lines is uniform we assume that the resistance R, capacitance C, inductance L and leakage G per unit length of the transmission line are constant.

To derive the required model equations,

We consider a small section of the line between x and $x+\Delta x$ and apply Kirchoff's laws to a circuit. Thus $R\Delta x$, $L\Delta x$, $C\Delta x$, and $G\Delta x$ are, respectively, the resistance, inductance, capacitance, and conductance of the section. The quantity $G\Delta x$ (where G is expressed in mhos or siemens) is a "virtual" resistance so that the power lost through it to the ground is equal to that due to leakage.



(Circuit equivalent to a small section of the wire)

By Kirchoff's 2nd law between A, D, we obtain

$$e(x,t) - e(x+\Delta x, t)$$
$$= R\Delta x\, i(x,t) + L\Delta x \frac{\partial i(x,t)}{\partial t} \longrightarrow (1)$$

Applying Kirchoff's first law at the node 'B', we have

$$i(x,t) - i(x+\Delta x, t) = C\Delta x \frac{\partial e(x+\Delta x, t)}{\partial t}$$
$$+ G\Delta x\, e(x+\Delta x, t)$$
$$\longrightarrow (2)$$

Dividing (1) and (2) by Δx we get.

$$\frac{e(x,t) - e(x+\Delta x, t)}{\Delta x} = R i(x,t) + L\frac{\partial i(x,t)}{\partial t}$$
$$\longrightarrow (3)$$

$$\frac{i(x,t) - i(x+\Delta x, t)}{\Delta x} = C\frac{\partial e(x+\Delta x, t)}{\partial t}$$
$$+ G\, e(x+\Delta x, t)$$
$$\longrightarrow (4)$$

Taking $Lt_{\Delta x \to 0}$ we get.

$$\lim_{\Delta x \to 0} \frac{e(x,t) - e(x+\Delta x, t)}{\Delta x} = R i(x,t) + L\frac{\partial i(x,t)}{\partial t} \longrightarrow (5)$$

$$\lim_{\Delta x \to 0} \frac{i(x,t) - i(x+\Delta x, t)}{\Delta x} = C\frac{\partial e(x, t)}{\partial t} + G\, e(x, t) \longrightarrow (6)$$

Eqs. (5) and (6) can also be rewritten as

$$\frac{\partial e(x,t)}{\partial x} = -Ri(x,t) - L\frac{\partial i(x,t)}{\partial t} \longrightarrow (7)$$

$$\frac{\partial i(x,t)}{\partial x} = -Ge(x,t) - C\frac{\partial e(x,t)}{\partial t} \longrightarrow (8)$$

Equation for $e(x,t)$ and $i(x,t)$ can be obtained by differentiating Eqs. (7) and (8) w.r.t. x ~~respectively~~ i.e. from (7) w.r.t x we have

$$\frac{\partial^2 e(x,t)}{\partial x^2} = -R\frac{\partial i(x,t)}{\partial x} - L\frac{\partial^2 i(x,t)}{\partial t \partial x} \longrightarrow (9)$$

Using (8) in (9) we get

$$\frac{\partial^2 e(x,t)}{\partial x^2} = \left[-R - L\frac{\partial}{\partial t}\right]\frac{\partial i(x,t)}{\partial x}$$

$$= -\left[R + L\frac{\partial}{\partial t}\right]\left[-Ge(x,t) - C\frac{\partial e(x,t)}{\partial t}\right]$$

$$= \left[R + L\frac{\partial}{\partial t}\right]\left[Ge(x,t) + C\frac{\partial e(x,t)}{\partial t}\right]$$

or

$$\boxed{e_{xx} = LC\,e_{tt} + (LG + RC)e_t + RGe} \longrightarrow (10)$$

Now diff (8) w.r.t x

$$\frac{\partial^2 i(x,t)}{\partial x^2} = -G\frac{\partial e(x,t)}{\partial x} - C\frac{\partial^2 e(x,t)}{\partial t\partial x}$$

$$= -\left(G + C\frac{\partial}{\partial t}\right)\frac{\partial e(x,t)}{\partial x}$$

Using (7) we get.

$$\frac{\partial^2 i(x,t)}{\partial x^2} = -\left(G + C\frac{\partial}{\partial t}\right)\left[-Ri(x,t) - L\frac{\partial i(x,t)}{\partial t}\right]$$

$$= \left(G + C\frac{\partial}{\partial t}\right)\left[Ri(x,t) + L\frac{\partial i(x,t)}{\partial t}\right]$$

$$\boxed{i_{xx} = LC\, i_{tt} + (LG + RC)\, i_t + RG\, i} \longrightarrow (11)$$

Equations (10) and (11) are known as the telegraphic eqs.

Special Cases

High Frequency limit

To analyze qualitatively this limit; consider the case

$$e(x,t) = A(x)\sin(\omega t + \phi_1) \longrightarrow (1)$$

$$i(x,t) = B(x)\cos(\omega t + \phi_2) \longrightarrow (2)$$

and $\omega \gg 1$.

We know that

$$\frac{\partial e(x,t)}{\partial x} = -R\,i(x,t) - L\frac{\partial i(x,t)}{\partial t} \longrightarrow (3)$$

$$\frac{\partial i(x,t)}{\partial x} = -G\,e(x,t) - C\frac{\partial e(x,t)}{\partial t} \longrightarrow (4)$$

(We have already derived these eqs)

Now $-L\dfrac{\partial i}{\partial t} = LB\omega\sin(\omega t + \phi_2) \longrightarrow (5)$

$L\omega$ = the impedance

Now in Eq. (3) when $\omega \gg 1$ then

$L\dfrac{\partial i(x,t)}{\partial t}$ is much larger than Ri

($L\dfrac{\partial i(x,t)}{\partial t}$ has effective coefficient $L\omega$

and the first term $Ri(x,t)$ has effective coefficient R).

Hence in this limit of ω, Eq. (3) can be approximated as

$$\boxed{e_x = -L\, i_t} \longrightarrow (6)$$

From (4) and (1)

$$-C\frac{\partial e}{\partial t} = -C\omega A(x)\cos(\omega t + \phi_1) \longrightarrow (7)$$

Now $C\frac{\partial e}{\partial t}$ has effective coefficient $C\omega$ and the first term in Eq. (4) has effective coefficient G).

Thence $C\frac{\partial e}{\partial t}$ is much larger than Ge and so (4) can be approximated as

$$\boxed{i_x = -C\, e_t} \longrightarrow (8)$$

Now differentiating Eq. (6) w.r.t x and (8) w.r.t t we obtain

$$e_{xx} = -L\, i_{xt} \longrightarrow (9)$$

$$i_{xt} = -C\, e_{tt} \longrightarrow (10)$$

Making use of (10) in (9) we obtain

$$e_{xx} = -L\left[-C\, e_{tt}\right]$$

$$\Rightarrow \boxed{e_{xx} = LC\, e_{tt}} \longrightarrow (11)$$

Now differentiation of (6) and (8) w.r.t t and x respectively yields

$$e_{xt} = -L\, i_{tt} \longrightarrow (12)$$

$$i_{xx} = -C\, e_{xt} \longrightarrow (13)$$

With (12) in (13) gives

$$i_{xx} = -C\left[-L\, i_{tt}\right]$$

$$\boxed{i_{xx} = LC\, i_{tt}} \longrightarrow (14)$$

We note that Eqs. (11) and (14) are the wave equations.

(It is of the form of $u_{xx} = \frac{1}{c^2} u_{tt}$) soln is $u(x,t) = f(x-ct) + g(x+ct)$ where f and g are arbitrary ftns).

With this in mind the solutions of (11) and (14) are

$$\boxed{\begin{aligned} e(x,t) &= f_1\left(x - \frac{1}{\sqrt{LC}}t\right) + g_1\left(x + \frac{1}{\sqrt{LC}}t\right). \\ i(x,t) &= f_2\left(x - \frac{1}{\sqrt{LC}}t\right) + g_2\left(x + \frac{1}{\sqrt{LC}}t\right). \end{aligned}} \longrightarrow (15)$$

Low-frequency limit.

In this case, i and e change very slowly with time. For this case $\omega \ll 1$.

Now from Eq. 3

The effective coefficient of second term $\left(L\frac{\partial i}{\partial t}\right)$ is $L\omega$ is much smaller than the effective coefficient of the first term (Ri) is R.

Thus (3) becomes now

$$\boxed{\frac{\partial e}{\partial x} = -Ri} \longrightarrow (16)$$

Consider Eq. (4)

Now for $\omega \ll 1$, the effective coefficient of $C\frac{\partial e}{\partial t}$ (i.e. $C\omega$) is much smaller than the effective coefficient of first term Ge (i.e G). Thus (4) in this limit of ω reduces to

$$\frac{\partial i}{\partial x} = -Ge \longrightarrow (17)$$

Now differentiate Eqs. (16) and (17) w.r.t. x we have

$$\frac{\partial^2 e}{\partial x^2} = -R\frac{\partial i}{\partial x} \longrightarrow (18)$$

$$\frac{\partial^2 i}{\partial x^2} = -G\frac{\partial e}{\partial x} \longrightarrow (19)$$

Making use of Eq. (17) in Eq. (18) and Eq. (16) into Eq. (19) we arrive at

$$\frac{\partial^2 e}{\partial x^2} = -R[-Ge]$$

$$\frac{\partial^2 i}{\partial x^2} = -G[-Ri]$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial^2 e}{\partial x^2} &= RGe \\ \frac{\partial^2 i}{\partial x^2} &= RGi \end{aligned}} \longrightarrow (20)$$

Submarine Cable

Earlier telecommunication signals between the United states and Europe were transmitted by cables that were laid down on the ocean floor. For these cables, $G \cong 0$ because of their extreme insulation. Moreover, the signal frequency ω is low. Under these circumstances we note from Eqs. (3) that the impedance $L\omega$ is much smaller than R. Hence, Eq. (3) yields

$$\frac{\partial e}{\partial x} = -Ri \longrightarrow (21)$$

Since $G = 0$, thus Eq. (4) reduces to

$$i_x = -C e_t \longrightarrow (22)$$

Diff. Eq. (21) w.r.t x we have

$$e_{xx} = -R\, i_x \longrightarrow (23)$$

using (22) in (23) we get

$$e_{xx} = -R\left[-c\, e_t\right]$$

$$\Rightarrow \boxed{e_{xx} = RC\, e_t} \longrightarrow (24)$$

Now differentiation of Eq. (22) gives

$$i_{xx} = -c\, e_{xt}$$

$$= -c\frac{\partial}{\partial t}\left(\frac{\partial e}{\partial x}\right) \quad \text{by (21)}$$

$$= -c\frac{\partial}{\partial t}(-Ri)$$

$$= RC\frac{\partial i}{\partial t}$$

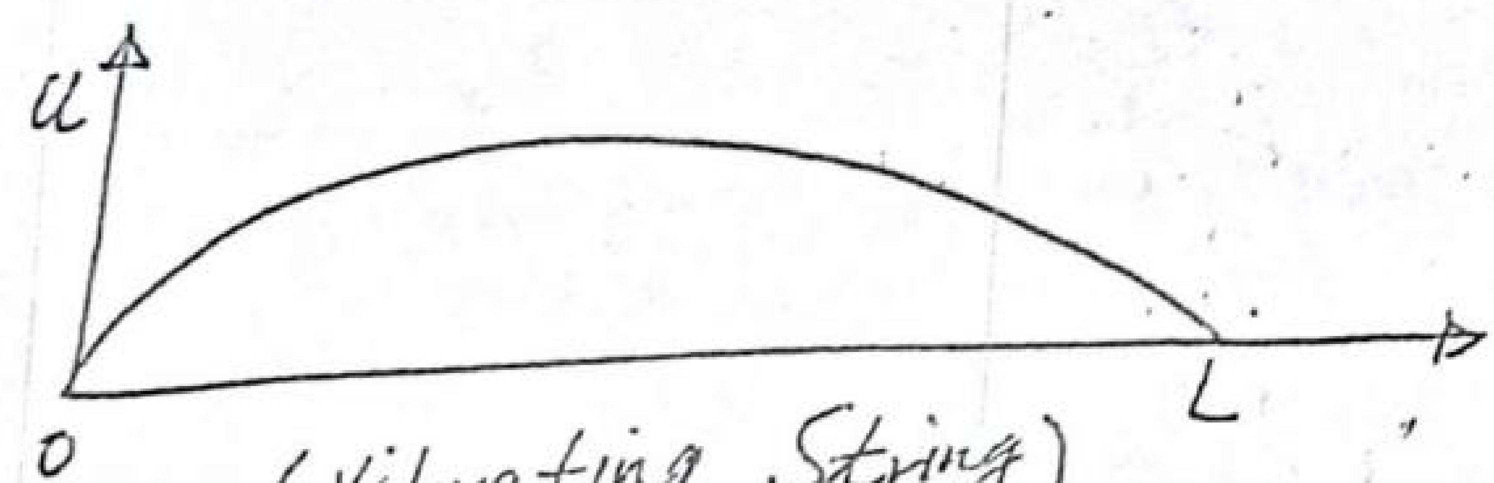
$$\Rightarrow \boxed{i_{xx} = RC\, i_t} \longrightarrow (25)$$

Note that: equations (24) and (25) are the one-dimensional diffusion equation. (These are parabolic equations).

Wave Phenomena

Objective

Construct a prototype mathematical model for the transverse vibrations of a string with fixed ends.



(Vibrating String)

Background

Generally, the wave phenomena requires considering the elastic properties of matter and leads to a complicated set of equations. To overcome this difficulty we make the following simplifying approximations and idealizations so that a prototype model can be constructed by applying only Newton's second law $F = m\bar{a}$ (i.e. the force equals the mass multiplied by the acceleration) to the system under study.

Approximations and Idealizations

1. The string is rigidly attached at its end points.
2. The string vibrates in one plane.
3. No external forces act on the string (prototype model)

(50)

4. The string does not suffer from damping forces.

5. The string is homogeneous. In particular, this implies that the linear density ρ and the mass per unit length m of the string are constant.

6. The deflection of the string from equilibrium and its slope are always small. Consequently, we are able to make the following two approximations:

a. The magnitude of the tension force $T(x,t)$ in the string is constant i.e. $T(x,t) = T$.

b. The string is rigid longitudinally i.e. a point on the string moves only in the vertical direction.

7. The tension force in the string is always tangential to it. This is usually expressed by saying that the string is assumed to be perfectly flexible.

Modeling

Consider a small segment of the string between x and $x + \Delta x$ as shown in Fig.

$$\bar{T} = -p\bar{I} + \bar{\sigma}$$

$$\bar{\sigma} = \underbrace{2\mu D}_{\text{viscosity}} + \bar{S}$$

$$S + \lambda \left[\quad \right] = 2\eta D.$$

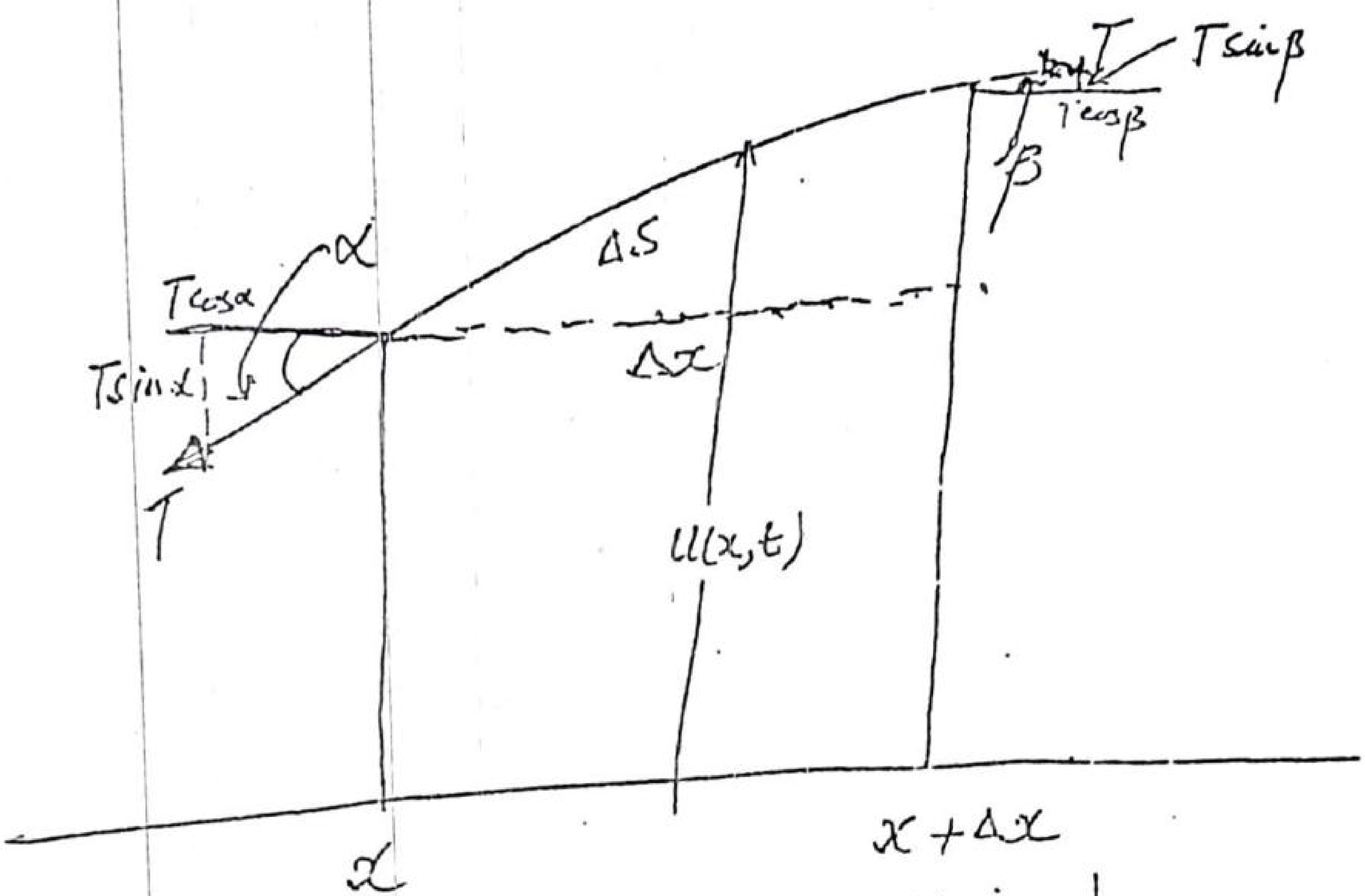


Fig: (Small segment of the string)

Before we can apply Newton's 2nd law of motion of this segment, we must make the following observations:

1. By approximation 6b, the segment is not moving in the horizontal direction.
2. The mass of the segment is $\rho \Delta s$. Since we are considering only small deflections, $|u| << 1$. It follows that $\Delta s \approx \Delta x$.
3. The acceleration of the segment in the vertical direction is given by $\left(\partial^2 u / \partial t^2\right)$ ($u(x,t)$ is the displacement in the vertical direction)
4. The sum of the vertical forces acting on the segment is $T\sin\beta - T\sin\alpha = T(\sin\beta - \sin\alpha)$.

By Newton's 2nd law

$$\bar{F} = m\bar{a}$$
$$= m\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

Since $m = \rho \Delta S \simeq \rho \Delta x \longrightarrow (2)$

using (2) in (1) we have

$$\bar{F} = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \longrightarrow (3)$$

From (4) $\bar{F} = T(\sin\beta - \sin\alpha) \longrightarrow (4)$

Making use of (3) in (4) we get

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(\sin\beta - \sin\alpha) \longrightarrow (5)$$

Since the deflection of the string and slope of the string is small (approximation 6) hence α, β are small and so

$$\sin\alpha \cong \tan\alpha = \frac{\partial u(x,t)}{\partial x} \longrightarrow (6)$$

$$\sin\beta \cong \tan\beta = \frac{\partial u}{\partial x}(x+\Delta x, t) \longrightarrow (7)$$

Substituting Eqs. (6) and (7) into Eq. (5) we obtain

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right]$$

Dividing by Δx we get

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right]}{\Delta x} \quad (b3)$$

Taking $\lim_{\Delta x \to 0}$ we get

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right]}{\Delta x}$$

$$= T \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

or $$\boxed{\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}} \longrightarrow (8)$$

where $c^2 = \frac{T}{\rho}$.

Note that Eq. (8) is the wave equation in one dimension.

Remark: From figure we note that the sum of the horizontal forces acting on the string segment is $T(\cos\beta - \cos\alpha) \neq 0$. Hence the segment must have an acceleration in the horizontal direction, which contradicts approximation 6b. However, since α, β are small, we can take $T(\cos\beta - \cos\alpha)$ is negligible.

Objective

Derive a model equation for the vibration of the string if a vertical external force $F(x,t)$ per unit length of the string acting on it.

Approximations and Idealizations

Here all the approximations and idealizations are the same as in previous article except approximation 3. Now, there is a vertical external force $F(x,t)$ per unit length.

Modeling

By 2nd law of motion $\bar{F} = m\bar{a}$

$$\approx \rho \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} \longrightarrow (1)$$

$$\text{vertical force} = T(\sin\beta - \sin\alpha) + F(x,t)\Delta x \longrightarrow (2)$$

$$\left.\begin{aligned} \sin\beta &\simeq \tan\beta = \frac{\partial u}{\partial x}(x+\Delta x, t) \\ \sin\alpha &\simeq \tan\alpha = \frac{\partial u}{\partial x}(x, t) \end{aligned}\right\} \longrightarrow (3)$$

(3) is valid for small α, β.

Substitution of Eq. (3) in (2) yields

vertical force = total force acting on the segment

$$= T\left[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u}{\partial x}\right] + F(x,t)\Delta x \longrightarrow (4)$$

From eqs. (1) and (4)

$$\rho \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} = T\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}\right] + F(x,t)\,\Delta x$$

Dividing by Δx we get.

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{T}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}\right] + F(x,t)$$

Taking $\lim\limits_{\Delta x \to 0}$ the above equation yields

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}\right]}{\Delta x} + F(x,t).$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} + \frac{F(x,t)}{\rho}$$

or

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{F(x,t)}{\rho}} \longrightarrow (5)$$

where $c^2 = T/\rho$ is the wave speed.

<u>Objective</u>

Derive a model equation for very small vibrations of a vertically suspended chain whose length is L and whose mass density per unit length ρ is constant.

(This is in the context of compounding of the first object in modeling wave phenomena article).

Approximations

1. Since the amplitude u of the vibrations is small, we assume that a point on the chain does not change its x-coordinate (see Fig. a)

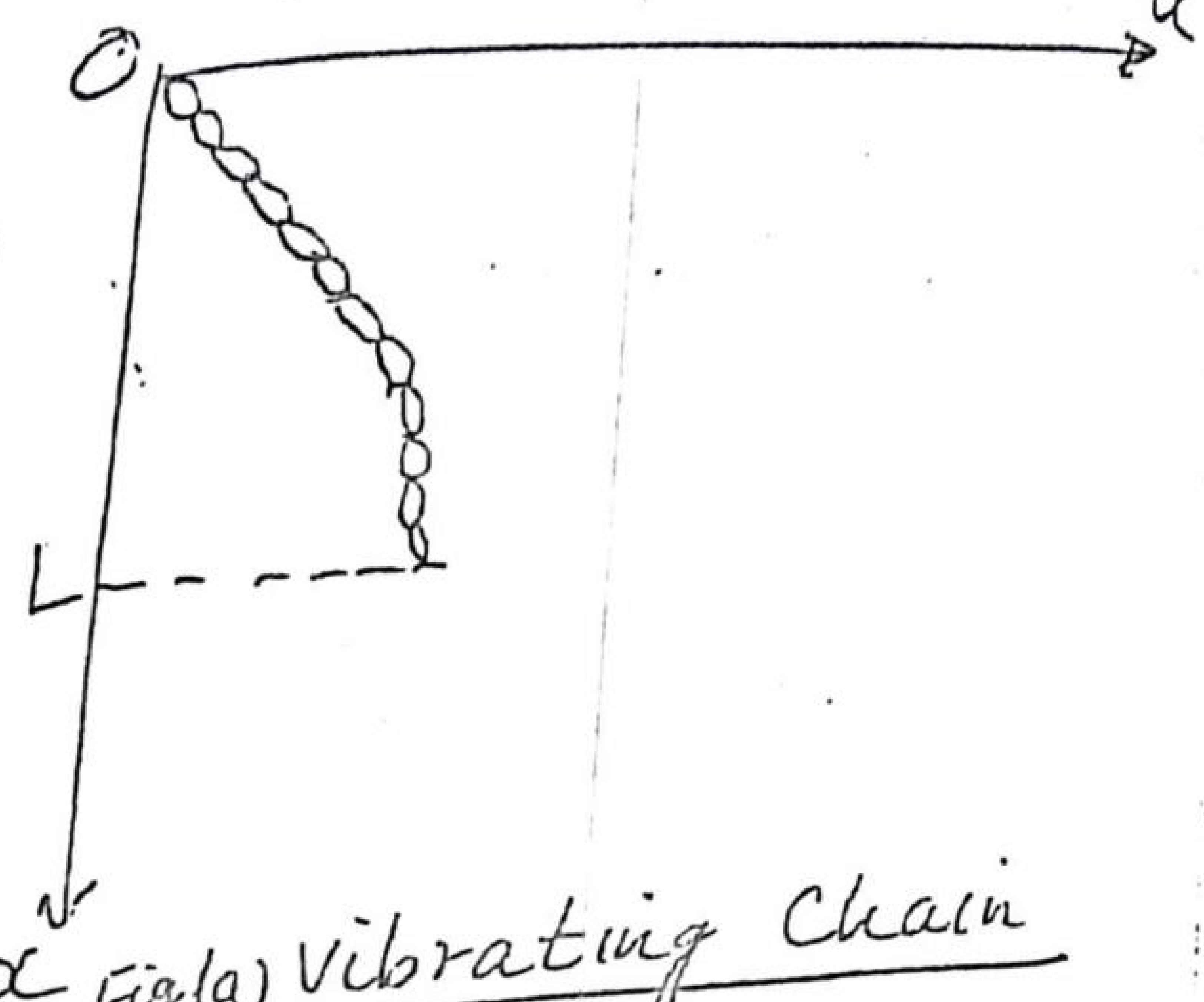


Fig (a) Vibrating Chain

2. The tension $T(x, t)$ in the chain cannot be assumed to be constant in the present situation. In fact, in the equilibrium (vertical) position of the chain

$$T(x) = \rho g(L-x) \qquad (1)$$

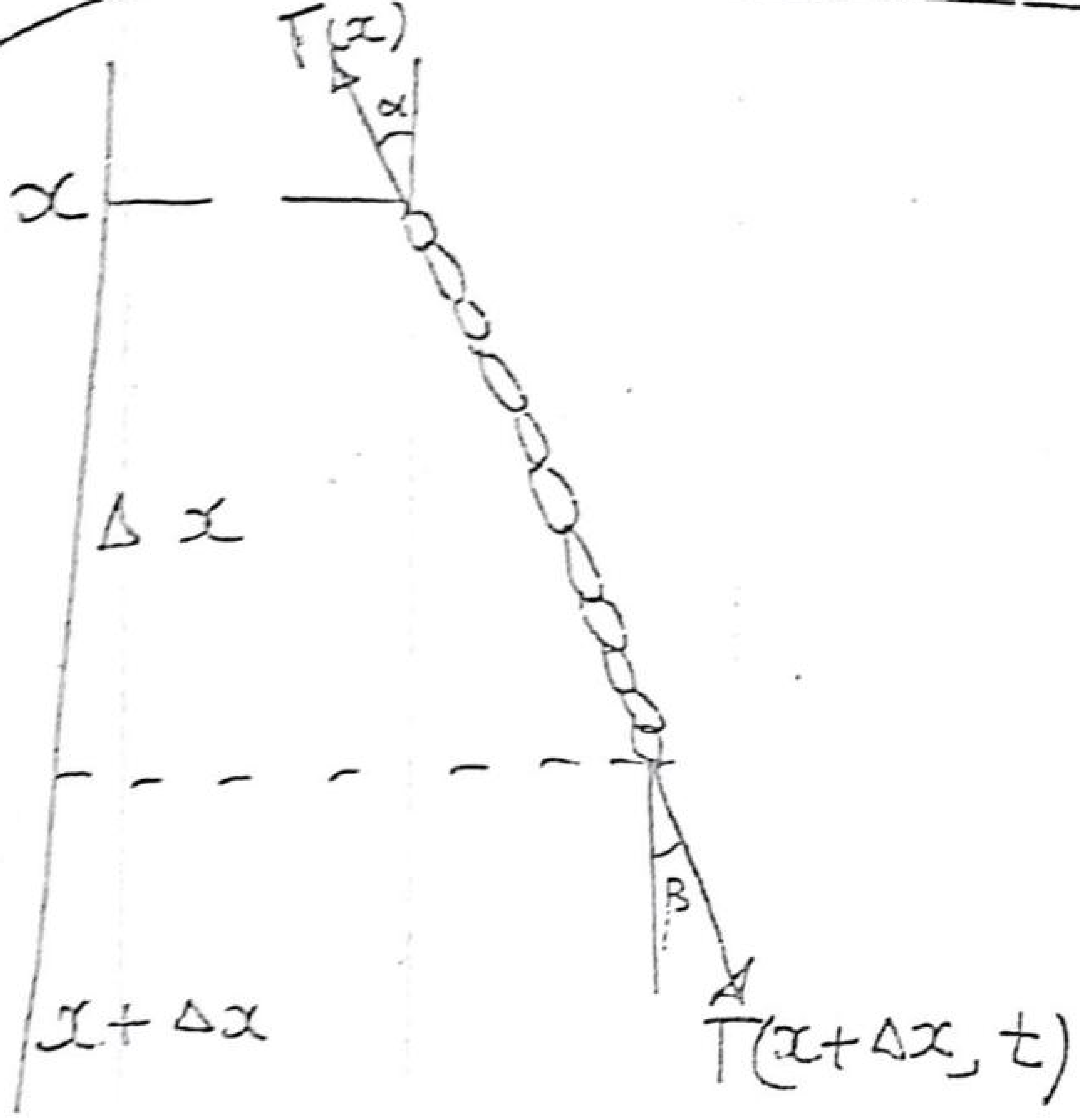
The above equation gives an acceptable approximation for the tension in the vibrating chain when $|u| << 1$ and $\left|\frac{\partial u}{\partial t}\right| << 1$.

3. Other approximations and idealizations of the prototype model remain intact.

Modeling

For the construction of mathematical model we once again consider a small section of chain between $[x, x+\Delta x]$. Applying Newtons

2nd law in the horizontal direction to such a section (see Fig. b) we have,



(Fig (b) small section of the vibrating chain)

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x)\sin\beta - T(x)\sin\alpha. \longrightarrow (2)$$

For small α, β.

$$\sin\beta \cong \tan\beta = \frac{\partial u(x+\Delta x, t)}{\partial x}$$

$$\sin\alpha \cong \tan\alpha = \frac{\partial u(x, t)}{\partial x}$$

Using above eqs. in (2) we arrive at

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x)\frac{\partial u(x+\Delta x, t)}{\partial x} - T(x)\frac{\partial u(x, t)}{\partial x}$$

or

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta x}\left[T(x+\Delta x)\frac{\partial u(x+\Delta x, t)}{\partial x} - T(x)\frac{\partial u(x, t)}{\partial x}\right]$$

Taking $\lim_{\Delta x \to 0}$ we have

$$\rho \frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[T(x+\Delta x) \frac{\partial u(x+\Delta x, t)}{\partial x} - T(x) \frac{\partial u}{\partial x}(x,t) \right]$$

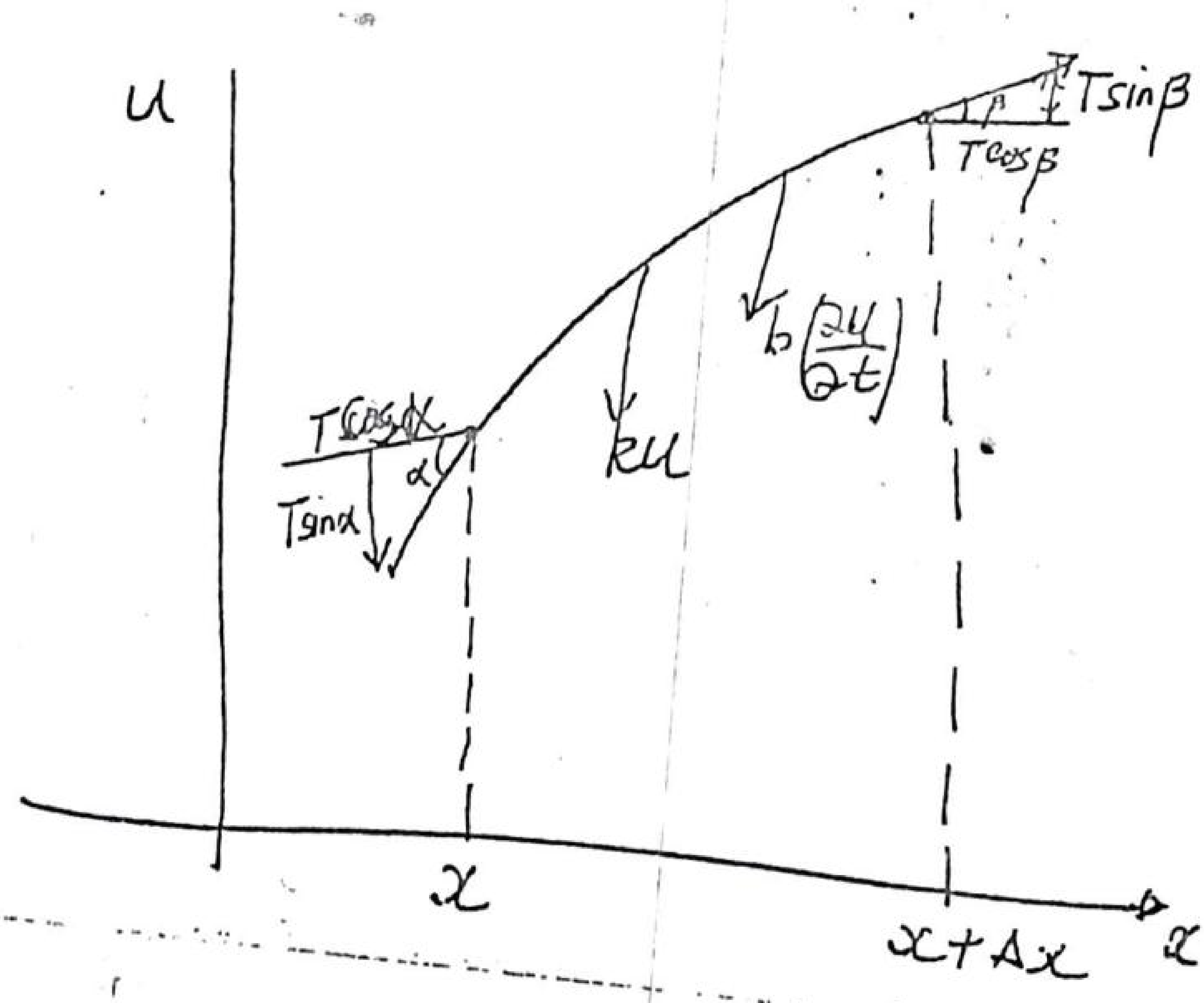
$$= \frac{\partial}{\partial x}\left[T(x) \frac{\partial u}{\partial x} \right] \longrightarrow (3)$$

Substituting (1) in (3) we obtain

$$\cancel{\rho} \frac{\partial^2 u}{\partial t^2} = \cancel{\rho} g \frac{\partial}{\partial x}\left[(L-x) \frac{\partial u}{\partial x} \right].$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = g \frac{\partial}{\partial x}\left[(L-x) \frac{\partial u}{\partial x} \right]}$$

Object: Derive a model equation for the vibration of the string when its motion is subject to an elastic restraint and a damping force.



Note: (*) The restraint force can be considered as a force of ku per unit length acting to return the string to its equilibrium position.

(**) The damping force is given by $b\left(\frac{\partial u}{\partial t}\right)$ per unit length and to oppose its motion.

$\boxed{\text{Modeling}}$

The restraint force per unit length $= -ku\Delta x$ ——— (1)

The damping force per unit length $= -b\left(\frac{\partial u}{\partial t}\right)\Delta x$ ——→ (2)

Total vertical force $= F$

$$= T(\sin\beta - \sin\alpha) - ku\Delta x - b\frac{\partial u}{\partial t}\Delta x \longrightarrow (3)$$

Since α and β are small so

$$\left.\begin{aligned} \sin\alpha &\simeq \tan\alpha = \frac{\partial u(x,t)}{\partial x} \\ \sin\beta &\simeq \tan\beta = \frac{\partial u(x+\Delta x,t)}{\partial x} \end{aligned}\right\} \longrightarrow (4)$$

Making use of Eq. (4) into Eq. (3) we have

$$\therefore F = T\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] - ku\Delta x - b\frac{\partial u}{\partial t}\Delta x$$

$$\longrightarrow (5)$$

By Newton's IInd law of motion

$$\bar{F} = m.\bar{a} = \rho \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} \longrightarrow (6)$$

From (5) and (6)

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]$$

$$- k u \Delta x - b \frac{\partial u}{\partial t} \Delta x$$

$\div$ing by Δx we get

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{T}{\Delta x}\left[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]$$

$$- k u - b \frac{\partial u}{\partial t}$$

Taking $\underset{\Delta x \to 0}{Lt}$ we get

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{\left[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]}{\Delta x}$$

$$- k u - b \frac{\partial u}{\partial t}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - k u - b \frac{\partial u}{\partial t}$$

$$\boxed{T \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} + k u + b \frac{\partial u}{\partial t}}$$

Objective

Build a model that describes the temperature distribution in a metal as a function of position and time. Consider the heat conduction problem in a rod of length L, made of homogeneous metal with constant cross sectional area A and rod is completely insulated along its lateral edges.

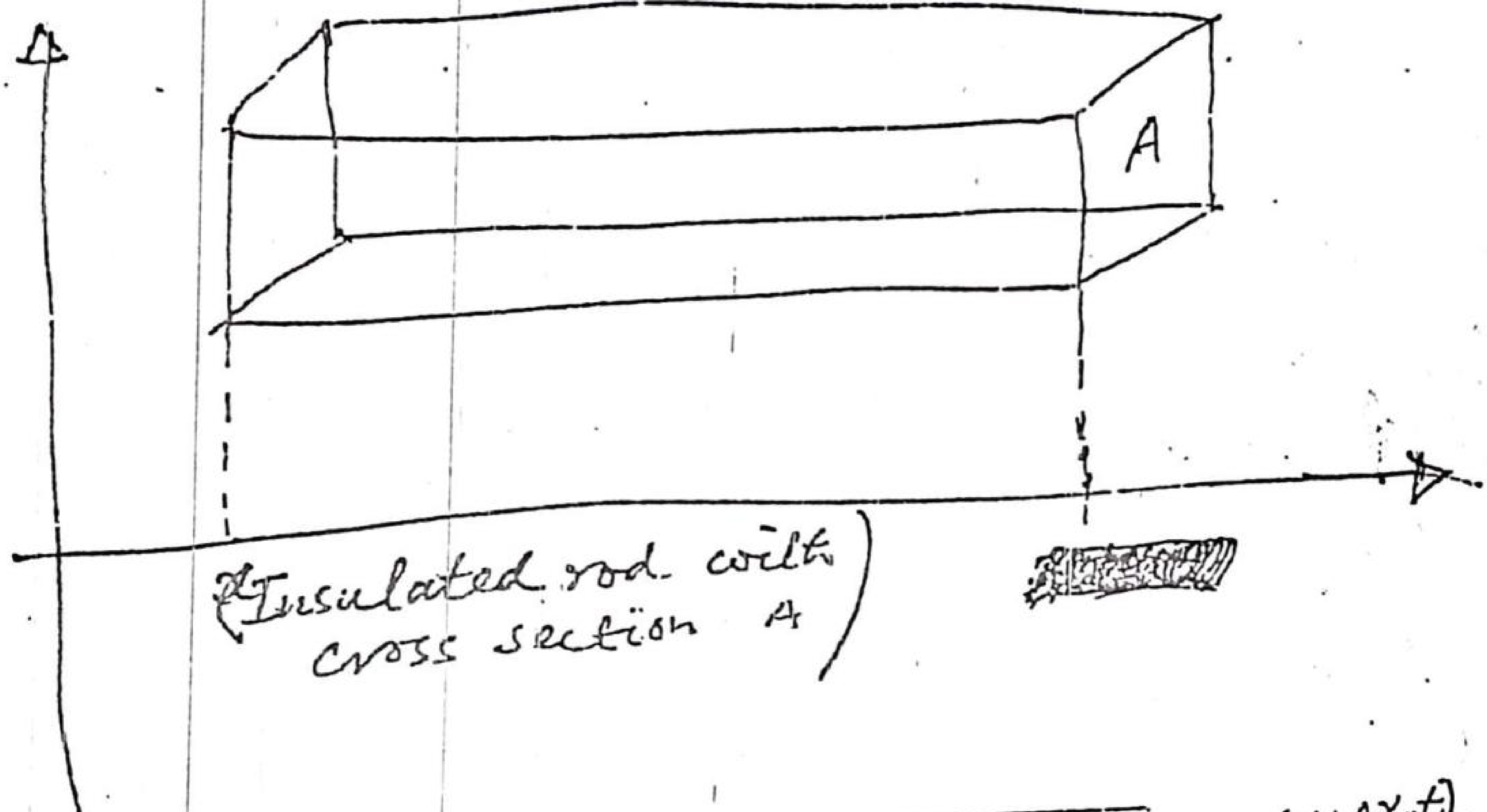
Approximations and idealizations

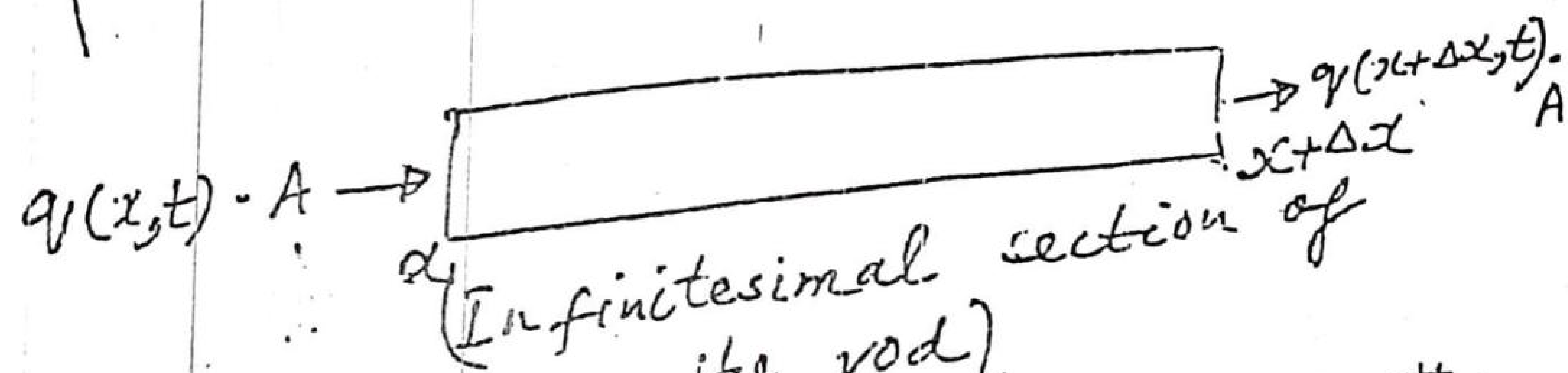
(i) We assume that rod is homogeneous. It follows that c, κ and ρ are independent of the position x. Also, we further assume for a prototype model that c, κ and ρ are independent of the temperature u.

(ii) The length of the rod remains constant inspite of the changes in its temperature.

(iii) We also assume that the rod is perfectly insulated along its lateral surface. Hence, heat can flow only in the horizontal direction, since a vertical flow will lead to heat accumulation along the edges, which is forbidden by the Fourier law of conduction. Therefore, we infer that the temperature on a vertical cross section of the rod must be the same. Thus, the temperature u depends only on x and t; that is $u = u(x, t)$.

(iv) We assume that heat flows in the rod from left to right, which requires the left side to be warmer than the right. (8)





We consider an infinitesimal element of the rod between x and $x+\Delta x$ and write the equation for energy conservation in it.

ΔV = Volume of the element $= A\,\Delta x$ —— (1)

where A is the cross sectional area of the element.

mass (Δm) of the element $= \rho A \Delta x = \rho \Delta V$ —— (2)

The amount of heat in the element at time t is $Q(x, t, \Delta x) = C \Delta m\, u(x,t)$ → (3)

$$Q(x,t,\Delta x) = c\rho A \Delta x\, u(x,t) \longrightarrow (4)(8.2)$$

The rate of change in $Q = \frac{dQ}{dt}$ $\longrightarrow (5)$

From (4) and (5)

$$\frac{dQ}{dt} = c\rho A \Delta x \frac{\partial u}{\partial t} \longrightarrow (6)$$

$$\left.\begin{array}{l}\text{Heat flowing in} = q(x,t)A \\ \text{Heat flowing out} = q(x+\Delta x,t)A.\end{array}\right\} \longrightarrow (7)$$

By principle of energy conservation,

"the rate of change must equal the rate at which heat is flowing in less the rate at which it is flowing out". Hence

$$\frac{dQ}{dt} = q(x,t)A - q(x+\Delta x,t)A \quad \text{(by 7)}$$

$$= A\left[q(x,t) - q(x+\Delta x,t)\right] \longrightarrow (8)$$

Substitution of Eq. (6) into Eq. (8) yields

$$c\rho \cancel{A} \Delta x \frac{\partial u}{\partial t} = \cancel{A}\left[q(x,t) - q(x+\Delta x,t)\right].$$

$$c\rho \frac{\partial u}{\partial t} = \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$$

Taking $\lim \Delta x \to 0$ we get.

$$c\rho \frac{\partial u}{\partial t} = \lim_{\Delta x \to 0} \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$$

$$= -\frac{\partial q}{\partial x} \longrightarrow (9)$$

By Fouriers law of heat conduction (in 1-dim) gives (8.3)

$$q = -K\left(\frac{\partial u}{\partial x}\right) \rightarrow (10)$$

Making use of (10) in (9) we have

$$c\rho \frac{\partial u}{\partial t} = +K \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{K} \frac{\partial u}{\partial t}$$

or $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}}$ $\Rightarrow$ (Heat eq. in 1-dim) or Diffusion eq. in 1-dim

where $k = \frac{K}{c\rho}$.

is thermal diffusivity.

Ex Generalize the prototype model for the case when heat is generated in the rod at a rate of $r(x,t)$ per unit volume.

Soln: Volume of the element $= A\,\Delta x$

$$\text{mass} = \Delta m = \rho A \Delta x \qquad \text{——— (1)}$$

Amount of heat in this element at time t

$$= Q(x,t,\Delta x) = c\,\Delta m\, u(x,t) \qquad \text{——— (2)}$$

Using (1) in (2) we obtain

$$Q = c\rho A\, u(x,t)\, \Delta x. \qquad \text{——— (3)}$$

Differentiation of above equation yields

$$\frac{dQ}{dt} = c\rho A \frac{\partial u}{\partial t} \Delta x \qquad \text{——— (4)}$$

By principle of energy conservation

$$\frac{dQ}{dt} = q(x,t)A - q(x+\Delta x,t)A + r(x,t)\,\underline{\underline{\Delta V}} \qquad (\text{as } \Delta V = A\Delta x)$$

$$= A\left[q(x,t) - q(x+\Delta x,t) + r(x,t)\Delta x\right] \qquad \text{——— (5)}$$

From (4) and (5)

$$c\rho \cancel{A} \frac{\partial u}{\partial t} \Delta \cancel{x} = \cancel{A}\left[\frac{q(x,t) - q(x+\Delta x,t)}{\Delta x} + r(x,t)\right]$$

$$c\rho\frac{\partial u}{\partial t} = -\left[\frac{q(x+\Delta x,t)-q(x,t)}{\Delta x}\right] + r(x,t) \quad (5)$$

Letting $\lim_{\Delta x \to 0}$ we get

$$c\rho\frac{\partial u}{\partial t} = -\lim_{\Delta x \to 0}\left[\frac{q(x+\Delta x,t)-q(x,t)}{\Delta x}\right] + r(x,t)$$

$$c\rho\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} + r(x,t) \longrightarrow (6)$$

By Fourier's law

$$q = -\kappa\nabla u$$

For 1-dim $q = -\kappa\dfrac{\partial u}{\partial x} \longrightarrow (7)$

Making use of Eq (7) in Eq. (6) we obtain

$$c\rho\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-\kappa\frac{\partial u}{\partial x}\right) + r(x,t)$$

$$= \kappa\frac{\partial^2 u}{\partial x^2} + r(x,t)$$

or

$$\frac{c\rho}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\kappa}r(x,t)$$

or

$$\boxed{\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\kappa}r(x,t)}$$

with $k = \dfrac{\kappa}{c\rho}$ is the thermal diffusivity

κ = Thermal Conductivity.

Note. 1-dim heat equation in absence of any internal sources for head in a rod of length L is $\frac{\partial u}{\partial t} = \frac{1}{k}\frac{\partial^2 u}{\partial x^2}$, $0 < x < L, t > 0$.

Initial condition

Since 1-dimensional heat equation is first order in t, it needs only one initial condition, which is normally taken to be

$$u(x,0) = f(x), \quad 0 < x < L.$$

This means prescribing the initial distribution of temperature in the rod.

Boundary conditions

The equation is of second order with respect to the space variable, so we need two boundary conditions. There are three main types of such conditions prescribed at the endpoints $x = 0$ and $x = L$ which have physical significance.

(i) The temperature may be at one end point; for example,

$$u(0,t) = \alpha(t), \quad t > 0.$$

(ii) If the rod is insulated at an endpoint, then the condition must mean that the heat flux there is zero. This is equivalent to the derivative $u_x = \frac{\partial u}{\partial x}$ (87) being equal to zero; for example

$$u_x(L,t) = 0, \quad t > 0.$$

More generally, if the heat flux through the end point $x = L$ is prescribed the above condition becomes

$$u_x(L,t) = \beta(t), \quad t > 0.$$

(iii) When one of the endpoints is in contact with another medium, we use Newton's law of cooling, which states that "heat flux at that endpoint is proportional to the difference between the temperature of the rod and the temperature of the external medium: for example,

$$K_0 u_x(0,t) = H[u(0,t) - U(t)], \quad t > 0,$$

where $U(t)$ is the (known) temperature of the external medium and $H > 0$ is the heat transfer coefficient. Owing to the convention concerning the direction of heat flux, at the other endpoint this type of condition becomes

$$-K_0 u_x(L,t) = H[u(L,t) - U(t)]; \quad t > 0$$

Remarks (i) Only one boundary condition is prescribed at each end point.

(ii) The boundary condition at $x=0$ may differ from that at $x=L$.

(iii) It is easily verified that the heat equation is linear.

(iv) The one-dimensional heat equation is the simplest example of a so-called parabolic equation.

Def: A partial differential equation (PDE) and the initial conditions (ICs) and boundary conditions (BCs) associated with it form as initial boundary value problem (IBVP). If only initial conditions or boundary conditions are present, then we have an initial value problem (IVP) or a boundary value problem (BVP), respectively.

Example(*): The initial boundary value problem (IBVP) modeling heat conduction in a one-dimensional uniform rod with sources, insulated lateral surface, and temperature prescribed at both endpoints is of the form

$$u_t(x,t) = k u_{xx}(x,t) + \underbrace{Q(x,t)}_{\text{Source term}}, \quad 0<x<L, \; t>0, \quad \text{(PDE)}$$

$$\left.\begin{array}{l} u(0,t) = \alpha(t) \\ u(L,t) = \beta(t) \end{array}\right\} t>0. \longrightarrow \text{(BCs)}$$

$$u(x,0) = f(x), \quad 0<x<L.$$

Where Q, α, β, and f are given functions.

<u>Example</u> If the near endpoint is insulated and far one is kept in a medium of constant zero temperature, and if the rod contains no sources then the corresponding IBVP is

$$u_t(x,t) = k u_{xx}(x,t), \quad 0<x<L, \; t>0 \quad \text{(PDE)}$$

$$u_x(0,t) = 0, \quad u_x(L,t) + h u(L,t) = \beta(t), \quad t>0, \quad \text{(BCs)}$$

$$u(x,0) = f(x), \quad 0<x<L. \quad \text{(IC)}$$

where β and f are given functions and h is a known positive constant.

<u>Def.</u> By a classical solution of an IBVP we understand a function $u(x,t)$ that satisfies pointwise the PDE, BCs and ICs everywhere in the region where the problem is formulated.

Note (i) If the functions α, β, and f are sufficiently smooth to ensure that u, u_t, u_x, and u_{xx} are continuous in G and up to the boundary of G including the two corner points, then the the initial boundary value problem (IBVP) in example (*) has at most one solution.

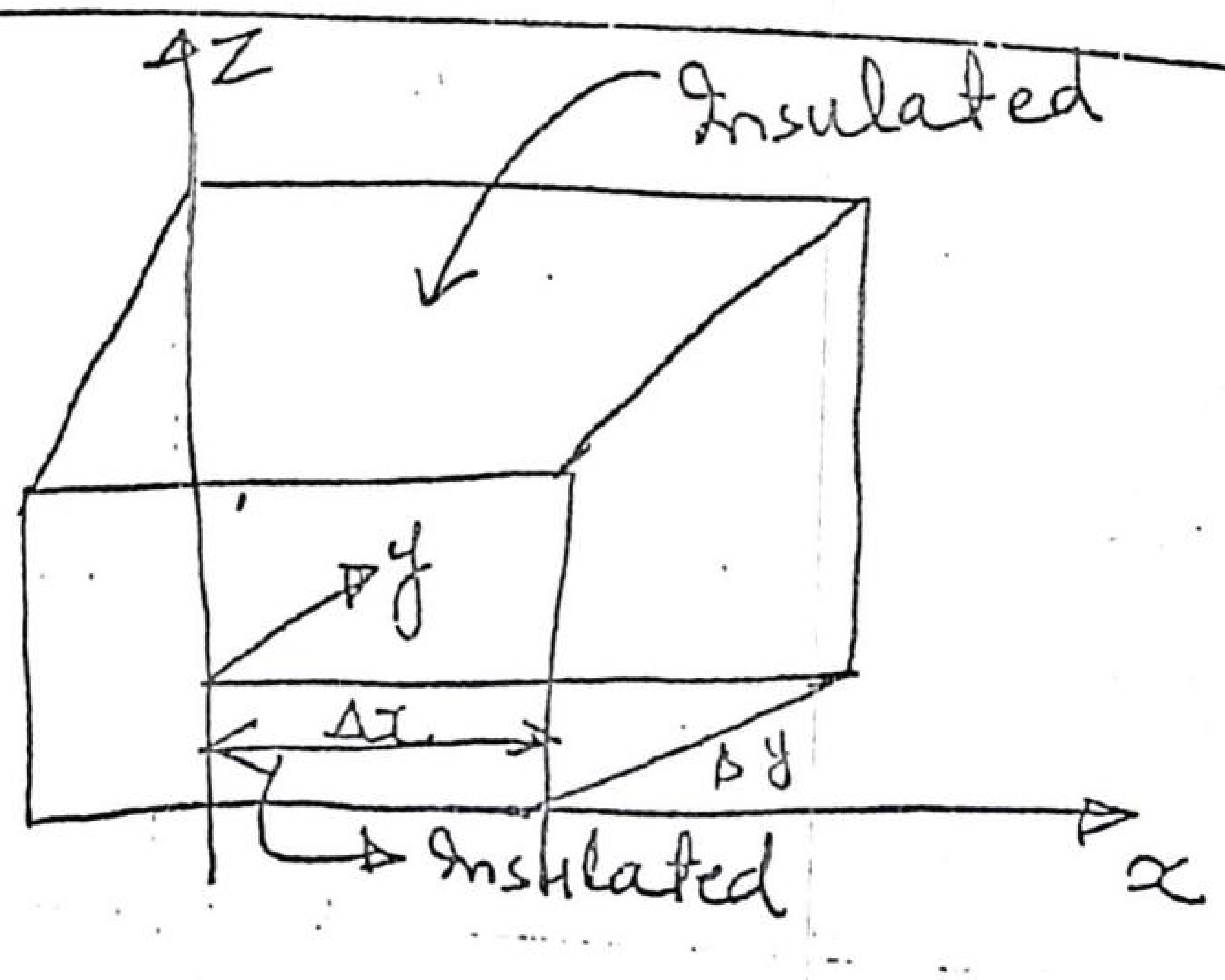
(ii) If a uniform rod is bent into a ring and the ends at $x=0$ and $x=L$ are joined. Then appropriate boundary conditions would be

$$u(0,t) = u(L,t), \quad t>0$$

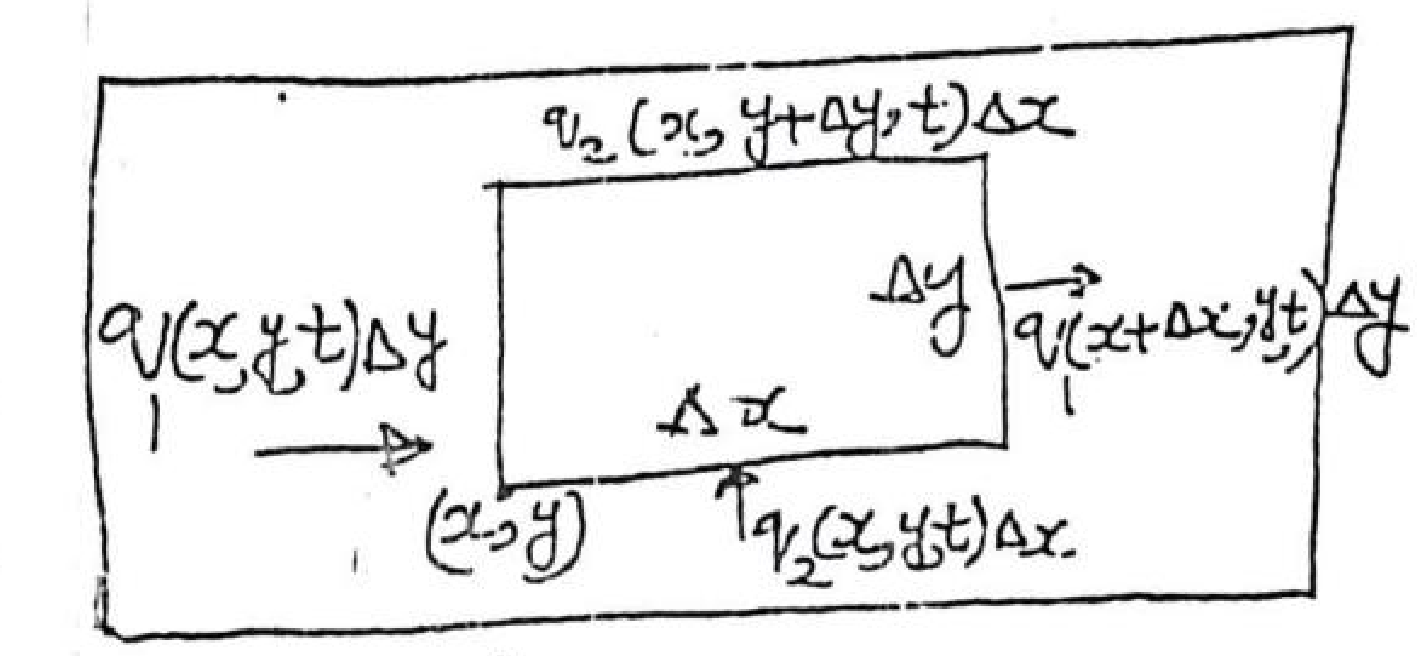
$$u_x(0,t) = u_x(L,t), \quad t>0$$

both of mixed type (BCs).

Heat or Diffusion Equation in two-dimension.



Consider a small rectangular element that is located at a point (x, y) in the plate.



(Heat balance in a two-dimensional element)

The volume of element $= \Delta x \Delta y\, h = A\Delta x = \Delta V$.
(A is the cross sectional area of the element).

Mass of the element $= m = \rho \Delta x \Delta y\, h$. —— ①
$= \rho \Delta V$.

The amount of heat Q in the element (at time t) $= c\, m\, u$ (by ①)

or

$$Q = c\, \rho\, \Delta x \Delta y\, h\; u(x, y, t) \longrightarrow ②$$

The rate of change in Q is $= \dfrac{dQ}{dt}$ $\longrightarrow$ ③

Making use of ② in ③ we obtain

$$\boxed{\frac{dQ}{dt} = c\rho\, \Delta x \Delta y\, h\, \frac{\partial u}{\partial t}} \longrightarrow ④$$

Now we decompose $\vec{q}$ as:

$$\vec{q} = q_1 \hat{i} + q_2 \hat{j} \quad (5)$$

and note that $q_1 \hat{i}$ is parallel to the boundary represented by the line between (x, y) and $(x+\Delta x, y)$. Thus $q_1 \hat{i}$ does not contribute to the flux through this boundary. A similar analysis holds for the other boundaries.

Now

Heat flows in the element

$$= q_1(x, y, t)\,\Delta y\, h + q_2(x, y, t)\,\Delta x\, h \quad (6)$$

$$\text{Heat flows out} = q_1(x+\Delta x, y, t)\,\Delta y\, h + q_2(x, y+\Delta y, t)\,\Delta x\, h. \quad (7)$$

By principle of energy conservation

$$\frac{dQ}{dt} = \text{Heat flow in} - \text{Heat flows out}$$

Using (6) and (7) we ~~get~~ obtain

$$\frac{dQ}{dt} = q_1(x, y, t)\,\Delta y\, h + q_2(x, y, t)\,\Delta x\, h - q_1(x+\Delta x, y, t)\,\Delta y\, h - q_2(x, y+\Delta y, t)\,\Delta x\, h.$$

or

(9)

$$\frac{dQ}{dt} = \left[q_1(x,y,t) - q_1(x+\Delta x, y, t)\right]\Delta y\, h$$
$$+ \left[q_2(x,y,t) - q_2(x, y+\Delta y, t)\right]\Delta x\, h$$

Making use of (4) in (8) we have $\rightarrow$ (8)

$$c\rho\, \Delta x \Delta y\, h \frac{\partial u}{\partial t} = \left[q_1(x,y,t) - q_1(x+\Delta x, y, t)\right]\Delta y h$$
$$+ \left[q_2(x,y,t) - q_2(x, y+\Delta y, t)\right]\Delta x h$$

$\div$ ing by $\Delta x \Delta y h$.

$$c\rho \frac{\partial u}{\partial t} = \frac{\left[q_1(x,y,t) - q_1(x+\Delta x, y, t)\right]}{\Delta x} + \frac{\left[q_2(x,y,t) - q_2(x, y+\Delta y, t)\right]}{\Delta y}.$$

Taking $\lim\limits_{\Delta x \to 0}$, $\lim\limits_{\Delta y \to 0}$ we obtain

$$c\rho \frac{\partial u}{\partial t} = \lim_{\Delta x \to 0} \frac{\left[q_1(x,y,t) - q_1(x+\Delta x, y, t)\right]}{\Delta x} + \lim_{\Delta y \to 0} \frac{\left[q_2(x,y,t) - q_2(x, y+\Delta y, t)\right]}{\Delta y}$$

$$= -\frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial y} = -\left(\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y}\right)$$

$$= -\operatorname{div} \bar{q}.$$

By Fourier law $\bar{q} = -k\bar{\nabla} u$.

as $\bar{\nabla}\cdot\bar{\nabla} u = \nabla^2 u$,

So $\boxed{c\rho \frac{\partial u}{\partial t} = k\nabla^2 u}$

or $\boxed{\frac{\partial u}{\partial t} = \frac{k}{c\rho}\nabla^2 u}$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

or
$$\frac{\partial u}{\partial t} = \frac{1}{k}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (x, y) \text{ in } G, \ t>0. \quad \longrightarrow (9)$$

This is the two-dimensional heat (diffusion) equation.

If the body contains sources; then (9) is replaced by its nonhomogeneous counterpart.

$$\boxed{\frac{\partial u(x,y,t)}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + Q(x,y,t), \quad (x,y) \text{ in } G, \ t>0.} \quad \longrightarrow (10)$$

Initial condition

Here this takes the form
$$u(x,y,0) = f(x,y), \quad (x,y) \text{ in } G,$$
and represents the initial distribution of temperature in the body.

Boundary conditions

The main types of BCs are similar in nature to those for a rod.

(i) when the temperature is prescribed on the boundary,
$$u(x,y,t) = \alpha(x,y,t), \quad (x,y) \text{ on } \partial D, \ t>0.$$

This is called a Dirichlet boundary condition. (∂ is the boundary).

(ii) If the flux through the boundary is prescribed, then

$$\nabla u(x,y,t)\cdot\vec{n}(x,y) = \beta(x,y,t),\ (x,y) \text{ on } \partial D,\ t>0,$$

or, equivalently,

$$u_n(x,y,t) = \beta(x,y,t),\ (x,y) \text{ on } \partial D,\ t>0,$$

where $\vec{n}$ is the unit outward normal to ∂D and $u_n = \frac{\partial u}{\partial n}$. This is called a Neumann boundary condition. In particular, when the boundary is insulated we have

$$u_n(x,y,t) = 0,\ (x,y) \text{ on } \partial D,\ t>0.$$

(iii) Newton's law of cooling takes the form

$$-K_0\ \nabla u(x,y,t)\cdot\vec{n}(x,y) = H\left[u(x,y,t) - U(x,y,t)\right],$$

$$x,y \text{ on } \partial G,\ t>0.$$

This is referred to as a Robin boundary condition.

Note: Sometimes we may have one type of condition prescribed on some part of the boundary, and another type on the remaining part.

Equilibrium temperature

An equilbrium (steady state) temperature is a time-independent solution $u = u(x, y)$ of (9). In other words, it is a function satisfying the Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad (x, y) \text{ in } G,$$

with a time-independent boundary condition. For example,

$$u(x, y) = \alpha(x, y), \quad (x, y) \text{ on } \partial D.$$

(*) If steady-state sources $Q(x, y)$ are present in the body then the equilibrium temperature satisfies the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{1}{k} Q(x, y),$$
$$(x, y) \text{ in } G.$$

Note: When we discuss the Poisson equation i.e. the non-homogeneous Laplace equation – we omit the factor $-1/k$ on the right-hand side, regarding it as incorporated in the source term Q.

(*) An equilbrium temperature may, or may not, exist.

Remarks

(i) Problems in three space variables are formulated similarly, with

$$u = u(x, y, z, t).$$

(ii) In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

where $u = u(r, \theta)$.

(iii) In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, z, the Laplacian takes the form

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

(iv) In circularly (axially) symmetric problems the function u is independent of θ, so

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}.$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$

(v) It is obvious that the Laplace and Poisson equations are linear.

(vi) The Laplace equation is the simplest example of a so-called elliptic equation.

<u>Example</u>: The equilibrium temperature distribution in a uniform finite plate with sources, insulated upper and lower faces, and prescribed temperature on the boundary is modeled by the BVP.

$$\nabla^2 u(x,y) = Q(x,y), \quad (x,y) \text{ in } G, \quad \text{(PDE)}$$

$$u(x,y) = \alpha(x,y), \quad (x,y) \text{ on } \partial G, \quad \text{(BC)}$$

where ∂G is a simple closed curve. The Laplacian is written either in Cartesian or in polar coordinates, depending on the geometry of the plate.

<u>Remark</u>: A (classical) solution of the BVP in above equation has the following properties:

(i) it is twice continuously differentiable in G and satisfies the PDE at every point in D;

(ii) it is continuous up to the boundary ∂D of D and satisfies the BC at every point of ∂D.

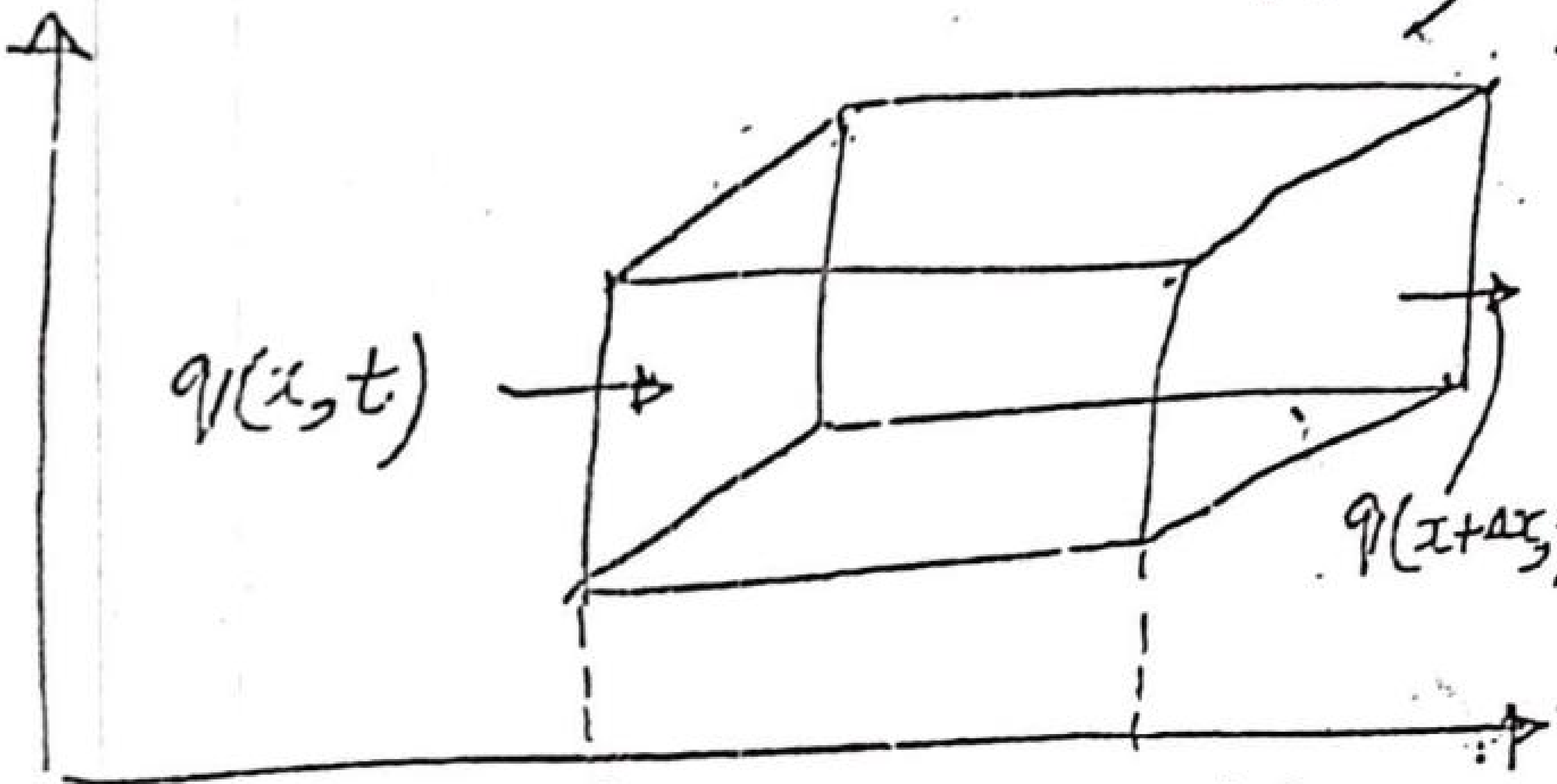
<u>Note</u> (*) If u is a solution of the IBVP in above example, then u attains its maximum and minimum values on the boundary ∂G of G.

(99)

(RK) If a solution u of the BVP in above example is identically zero on ∂G, then u is also identically zero in G.

Ex Discuss the prototype problem when c, K and ρ depend on x (i.e. non-homogeneous rod)

Soln



Let us take an infinitesimal element of the rod between x and $x+\Delta x$. The given data in usual notation is

$$c = c(x),\ \rho = \rho(x),\ K = K(x) \longrightarrow (1)$$

(as rod is nonhomogeneous)

If A is the cross-sectional area then

Volume of element $= A\Delta x = \Delta V$

mass " " $= m = \rho A \Delta x \longrightarrow (2)$

The amount of heat in the element at time $t = Q(x, t, \Delta x)$

$$= c(x)\, m\, u(x,t)$$

$$= c(x)\,\rho(x)\, A\, \Delta x\, u(x,t) \longrightarrow (3)$$

The rate of change in $Q = \dfrac{dQ}{dt} \longrightarrow (4)$

using Eq. (3) in Eq. (4) we get

$$\frac{dQ}{dt} = \frac{d}{dt}\left[c(x)\,\rho(x)\,A\,\Delta x\,u(x,t)\right]$$

$$= c\rho A\Delta x\frac{\partial u}{\partial t} + \frac{\partial c}{\partial t}\rho A\Delta x\,u(x,t)$$

$$+ c\frac{\partial \rho}{\partial t}A\Delta x\,u(x,t)$$

$$= c\rho A\Delta x\frac{\partial u}{\partial t} + \left(\frac{\partial c}{\partial x}\frac{\partial x}{\partial t}\right)\rho A\Delta x\,u(x,t)$$

$$+ c\left(\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t}\right)A\Delta x\,u(x,t) \longrightarrow (5)$$

By principle of conservation of energy

$$\frac{dQ}{dt} = A\left[q(x,t) - q(x+\Delta x,t)\right]$$

From Eqs. (5) and (6) we can write $\longrightarrow$ (6)

$$c\rho\cancel{A}\cancel{\Delta x}\frac{\partial u}{\partial t} + \left(\frac{\partial c}{\partial x}\frac{\partial x}{\partial t}\right)\rho\cancel{A}\cancel{\Delta x}\,u(x,t)$$

$$+ c\left(\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t}\right)\cancel{A}\cancel{\Delta x}\,u(x,t) = \cancel{A}\left[\frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}\right]$$

Letting $\Delta x \to 0$ we get.

$$c\rho\frac{\partial u}{\partial t} + \left(\frac{\partial c}{\partial x}\cdot\frac{\partial x}{\partial t}\right)\rho\,u(x,t)$$

$$+ c\left(\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t}\right)u(x,t) = \lim_{\Delta x \to 0}\frac{\left[q(x,t) - q(x+\Delta x,t)\right]}{\Delta x}$$

$$= -\frac{\partial q}{\partial x} \longrightarrow (7)$$

By Fourier's law we have

$$q = -K(x)\,\bar{\nabla} u$$

For 1-dim

$$q = -K(x)\frac{\partial u}{\partial x}.$$

$$\frac{\partial q}{\partial x} = -\frac{\partial K}{\partial x}\frac{\partial u}{\partial x} - K\frac{\partial^2 u}{\partial x^2}.$$

$$\boxed{-\frac{\partial q}{\partial x} = \frac{\partial K}{\partial x}\frac{\partial u}{\partial x} + K\frac{\partial^2 u}{\partial x^2}} \rightarrow (8)$$

With (7) and (8) we get

$$\boxed{c\rho\frac{\partial u}{\partial t} + \left(\frac{\partial c}{\partial x}\frac{\partial x}{\partial t}\right)\rho u + c\left(\frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t}\right)u = \frac{\partial K}{\partial x}\frac{\partial u}{\partial x} + K\frac{\partial^2 u}{\partial x^2}.}$$

(This is the resulting equation for a non-homogeneous rod)

6/10/03